Journal of Statistical Physics, Vol. 102, Nos. 3/4, 2001

Analytic Properties of the Structure Function for the One-Dimensional One-Component Log–Gas[1]

P. J. Forrester,[2] B. Jancovici,[3] and D. S. McAnally[2]

Received April 11, 2000

The structure function $S(k;\beta)$ for the one-dimensional one-component log–gas is the Fourier transform of the charge–charge, or equivalently the density–density, correlation function. We show that for $|k| < \min(2\pi\rho, 2\pi\rho\beta)$, $S(k;\beta)$ is simply related to an analytic function $f(k;\beta)$ and this function satisfies the functional equation $f(k;\beta) = f(-2k/\beta; 4/\beta)$. It is conjectured that the coefficient of k^j in the power series expansion of $f(k;\beta)$ about $k=0$ is of the form of a polynomial in $\beta/2$ of degree j divided by $(\beta/2)^j$. The bulk of the paper is concerned with calculating these polynomials explicitly up to and including those of degree 9. It is remarked that the small k expansion of $S(k;\beta)$ for the two-dimensional one-component plasma shares some properties in common with those of the one-dimensional one-component log–gas, but these break down at order k^8.

KEY WORDS: Logarithmic potential; two-dimensional plasma; fractional statistics; random matrices; exact solution.

1. INTRODUCTION

The one-component log–gas, consisting of N unit charges on a circle of circumference length L interacting via the two-dimensional Coulomb potential $\Phi(\vec{r}, \vec{r}\,') = -\log|\vec{r} - \vec{r}\,'|$, is specified by the Boltzmann factor

$$A_{N,\beta} \prod_{1 \leqslant j < k \leqslant N} |e^{2\pi i x_k/L} - e^{2\pi i x_j/L}|^{\beta}, \qquad 0 \leqslant x_j \leqslant L \tag{1.1}$$

[1] Dedicated to R. J. Baxter on the occasion of his 60th birthday.

[2] Department of Mathematics and Statistics, University of Melbourne, Parkville, Victoria 3052, Australia.

[3] Laboratoir de Physique Théorique, Université Paris-Sud, 91405 Orsay Cedex, France. Unité Mixte de Recherche 8627 (CNRS).

0022-4715/01/0200-0737$19.50/0

The constant $A_{N,\beta}$, which plays no role in the calculation of distribution functions, results from scaling the radius of the circle out of the logarithmic potential, and also includes the particle-background and background-background interactions (a uniform neutralizing background is imposed for thermodynamic stability). The thermodynamic limit $N, L \to \infty$, $N/L = \rho$ (fixed) is taken, which gives an infinite system on a straight line with particle density ρ. This system was first studied because of its relation to the theory of random matrices.[20] The thermodynamic functions were obtained. The pressure P has the simple form

$$\beta P = [1 - (\beta/2)]\, \rho \tag{1.2}$$

at any inverse temperature β. However, exact (simple) forms for the correlation functions were obtained by the pioneers only for the special temperatures corresponding to $\beta = 1, 2, 4$ (See Section 5). More recently, exact expressions for the two-body density were derived for arbitrary even integer β[3] and then for arbitrary rational β.[11] Unfortunately, these latter exact expressions are complicated multivariable integral representations which cannot be easily used as such for actual computations. The purpose of the present paper is to obtain explicit small k expansions for the structure function (the Fourier transform of the two-body density).

The log–gas is an example of a system interacting via the d-dimensional Coulomb system (here $d = 2$) but confined to a domain of dimension $d - 1$. It therefore exhibits universal features—that is features independent of microscopic details such as any short range potential between charges or the number of charge species—characteristic of Coulomb systems in this setting.[14] One universal feature is the existence of an algebraic tail in the leading non-oscillatory term of the large-distance asymptotic expansion of the charge–charge correlation function. For general charged systems in their conductive phase, interacting via the two-dimensional Coulomb potential in a one-dimensional domain, this is predicted to have the form[7]

$$-\frac{1}{\beta(\pi r)^2} \tag{1.3}$$

where r is the distance. For the one-component log–gas, (1.3) can be verified for all β rational.[8]

The verification is possible because the charge–charge correlation function (which for a onecomponent system is the same as the density–density correlation) is known explicitly for β rational[11] (see (2.3) below). In this work we further analyze the properties of the structure factor $S(k; \beta)$ (Fourier transform of the charge–charge correlation) for the one-component log–gas. In particular we are interested in the β dependence of the coefficients in the small k expansion of $S(k; \beta)$.

The large distance behaviour (1.3) is equivalent to the small k behaviour

$$S(k; \beta) \sim \frac{|k|}{\pi\beta} \tag{1.4}$$

Furthermore, by making use of the equivalence of the charge–charge and density–density correlation in the one-component log–gas, together with the exact equation of state the second order term in (1.4) has been predicted for general β,[6] giving

$$S(k; \beta) \sim \frac{|k|}{\pi\beta} + \frac{(\beta/2 - 1)\, k^2}{(\pi\beta)^2\, \rho} + O(|k|^3) \tag{1.5}$$

Let

$$f(k; \beta) := \frac{\pi\beta}{|k|}\, S(k; \beta), \qquad 0 < k < \min(2\pi\rho, \pi\beta\rho) \tag{1.6}$$

and define f for $k < 0$ by analytic continuation (we will see below that $f(k; \beta)$ is analytic for $0 \leqslant |k| < \min(2\pi\rho, \pi\beta\rho)$). In Section 2 we use the exact result (2.3) below to derive the functional equation

$$f(k; \beta) = f\left(-\frac{2k}{\beta}; \frac{4}{\beta}\right) \tag{1.7}$$

The simplest structure consistent with (1.7) is

$$\frac{\pi\beta}{|k|}\, S(k; \beta) = 1 + \sum_{j=1}^{\infty} p_j(\beta/2) \left(\frac{|k|}{\pi\beta\rho}\right)^j, \qquad |k| < \min(2\pi\rho, \pi\beta\rho) \tag{1.8}$$

where $p_j(x)$ is a polynomial of degree j which satisfies the functional relation

$$p_j(1/x) = (-1)^j\, x^{-j} p_j(x) \tag{1.9}$$

Equivalently, (1.9) can be stated as requiring

$$p_j(x) = \sum_{l=0}^{j} a_{j,\,l} x^l, \qquad a_{j,\,l} = a_{j,\,j-l} \quad (j \text{ even}) \tag{1.10}$$

$$p_j(x) = (x-1) \sum_{l=0}^{j-1} \tilde{a}_{j,\,l} x^l, \qquad \tilde{a}_{j,\,l} = \tilde{a}_{j,\,j-1-l} \quad (j \text{ odd}) \tag{1.11}$$

Inspection of (1.5) shows that the conjectured structure (1.8) is correct at order $|k|$ and furthermore gives

$$p_1(x) = (x-1) \tag{1.12}$$

and thus $\tilde{a}_{1,0} = 1$ in (1.11). In Section 3 we use (2.3) to verify that the structure (1.8) is correct at order k^2 and we compute $p_2(x)$ explicitly. In Section 4 we use an exact evaluation of the two-particle distribution function for β even[3] to rederive the result of Section 3, and we also use this formula to verify the structure (1.8) at order k^4 and to compute $p_4(x)$ explicitly.

Assuming the validity of (1.8) we see that $p_j(x)$ can be computed from knowledge of the coefficient of $|k|^j$ in $S(k; \beta)$, or the coefficient of $|k|^j$ in $\partial^p S(k; \beta)/\partial \beta^p$ ($p \leqslant j$), for an appropriate number of distinct values of β. Because the functional relation (1.7) has via (1.10) and (1.11) been made a feature of (1.8) the values of 3 cannot be related by $\beta \mapsto 4/\beta$. In Section 5 the known exact evaluation of $S(k; \beta)$ to leading order in β is reviewed, as are the exact evaluations of $S(k; 2)$ and $S(k; 4)$. Also noted are the exact evaluations of $S(k, 1)$ and $S(k; \beta)$ to leading order in $1/\beta$, which according to (1.7) are related to $S(k; 4)$ and $S(k; \beta)$ to leading order in β respectively. All of these exact evaluations are in terms of elementary functions, and so can be expanded to all orders in k. We then present the exact evaluation of $\partial S(k; \beta)/\partial \beta$ to leading order in β, as well as the exact evaluation of $\partial S(k; \beta)/\partial \beta$ evaluated at $\beta = 2$ and $\beta = 4$. The details of the latter two calculations are given in separate appendices. Again the final expressions can be expanded to high order in $|k|$. Using this data all polynomials in the expansion (1.8) up to and including the term with $j = 9$ can be computed. This expansion is written out explicitly in the final section and some special features of the polynomials therein, relating to the sign of the coefficients and the zeros, are noted. A physical interpretation of the functional equation, based on an analogy with a quantum many body system, which identifies an equivalence between quasi-hole and quasi-particle states contributing to $S(k; \beta)$ for $|k|$ small enough is given. We end with some remarks on the possible occurence of a functional equation analogous to (1.7) in the two-dimensional one-component plasma.

2. THE FUNCTIONAL EQUATION

The Boltzmann factor (1.1) also has the physical interpretation as the absolute value squared of the exact ground state wave function, $|0\rangle$ say, for the Calogero–Sutherland quantum many body Hamiltonian

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \beta(\beta/2 - 1)\left(\frac{\pi}{L}\right)^2 \sum_{1 \leqslant j < k \leqslant N} \frac{1}{\sin^2 \pi(x_j - x_k)/L} \tag{2.1}$$

This Hamiltonian describes quantum particles on a circle of circumference length L interacting via the inverse square of the distance between the particles. In the thermodynamic limit $N, L \to \infty$, $N/L = \rho$ (fixed) the N particle system becomes an infinite system on a line with particle density ρ. The ground state dynamical density–density correlation function

$$\rho^{\text{dyn.}}(0, x; t) := \langle 0 | \, n(0) \, e^{-iHt} n(x) \, e^{iHt} \, | 0 \rangle, \qquad n(y) := \sum_{j=1}^{N} \delta(y - x_j) \quad (2.2)$$

in the infinite system has been calculated exactly for all rational β.[11] The fact that $(|0\rangle)^2$ is proportional to (1.1) tells us that at $t = 0$ (2.2) is equal to

$$\rho^T_{(2)}(0, x) + \rho\delta(x)$$

where $\rho^T_{(2)}$ is the truncated two-body density, for the log–gas system. Thus the exact evaluation of

$$S(k; \beta) := \int_{-\infty}^{\infty} (\rho^T_{(2)}(0, x) + \rho\delta(x)) \, e^{ikx} \, dx$$

for the log–gas follows from the exact evaluation of (2.2) for the quantum system. Taking β to be rational and setting

$$\beta/2 := p/q \; =: \lambda$$

where p and q are relatively prime integers, the latter exact result gives[6]

$$S(k; \beta) = \pi C_{p,q}(\lambda) \prod_{i=1}^{q} \int_0^{\infty} dx_i \prod_{j=1}^{p} \int_0^1 dy_j \, Q^2_{p,q} F(q, p, \lambda \,|\, \{x_i, y_j\}) \, \delta(k - Q_{p,q}) \tag{2.3}$$

where

$$\begin{aligned}
C_{p,q}(\lambda) \; &:= \frac{\lambda^{2p(q-1)} \Gamma^2(p)}{2\pi^2 p! \, q!} \frac{\Gamma^q(\lambda) \, \Gamma^p(1/\lambda)}{\left(\begin{array}{l} \prod_{i=1}^{q} \Gamma^2(p - \lambda(i-1)) \\ \quad \times \prod_{j=1}^{p} \Gamma^2(1 - (j-1)/\lambda) \end{array} \right)} \\
Q_{p,q} \; &:= 2\pi\rho \left(\sum_{i=1}^{q} x_i + \sum_{j=1}^{p} y_j \right) \qquad\qquad (2.4) \\
F(q, p, \lambda \,|\, \{x_i, y_j\}) \; &:= \frac{\prod_{i<i'} |x_i - x_{i'}|^{2\lambda} \prod_{j<j'} |y_j - y_{j'}|^{2/\lambda}}{\prod_{i=1}^{q} \prod_{j=1}^{p} (x_i + \lambda y_j)^2} \\
&\quad \times \frac{1}{\prod_{i=1}^{q} (x_i(x_i + \lambda))^{1-\lambda} \prod_{j=1}^{p} (\lambda y_j(1 - y_j))^{1-1/\lambda}}
\end{aligned}$$

In the domain of integration of (2.3) the integration variables are all positive and because of the delta function are restricted to the hyperplane

$$\sum_{i=1}^{q} x_i + \sum_{j=1}^{p} y_j = \frac{|k|}{2\pi\rho}$$

We see immediately from these constraints that the restriction $y_j < 1$ in the domain of integration is redundant for

$$|k| < 2\pi\rho \tag{2.5}$$

Thus assuming (2.5) we can extend the integration over y_j to the region $(0, \infty)$. Doing this and changing variables $x_i \mapsto |k|\, x_i$ and $y_j \mapsto |k|\, y_j$ we see that for $|k|$ in the region (2.5)

$$\begin{aligned} S(k;\beta) = \pi\, |k|\, C_{p,q}(\lambda) \prod_{i=1}^{q} \int_0^\infty dx_i \prod_{j=1}^{p} \int_0^\infty dy_j\, Q_{p,q}^2 \\ \times \hat{F}(q, p, \lambda \mid \{x_i, y_j\}; k)\, \delta(1 - Q_{p,q}) \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} \hat{F}(q, p, \lambda \mid \{x_i, y_j\}; k) = \frac{1}{\prod_{i=1}^{q} (x_i(1 + kx_i/\lambda))^{1-\lambda} \prod_{j=1}^{p} (y_j(1 - ky_j))^{1-1/\lambda}} \\ \times \frac{\prod_{i<i'} |x_i - x_{i'}|^{2\lambda} \prod_{j<j'} |y_j - y_{j'}|^{2/\lambda}}{\prod_{i=1}^{q} \prod_{j=1}^{p} (x_i + \lambda y_j)^2} \end{aligned} \tag{2.7}$$

Notice that (2.7) is such that the integral in (2.6) is analytic for

$$|k| < \min(2\pi\rho, \pi\rho\beta) \tag{2.8}$$

Thus according to the definition (1.6) we read off that

$$\begin{aligned} f(k;\beta) = 2\pi^2 \lambda C_{p,q}(\lambda) \prod_{i=1}^{q} \int_0^\infty dx_i \prod_{j=1}^{p} \int_0^\infty dy_j\, Q_{p,q}^2 \\ \times \hat{F}(q, p, \lambda \mid \{x_i, y_j\}; k)\, \delta(1 - Q_{p,q}) \end{aligned} \tag{2.9}$$

The functional equation (1.7) is a simple consequence of this exact formula. Thus we see that the integral in (2.9) is unchanged by the mapping $\lambda \mapsto 1/\lambda$ (and thus $p \leftrightarrow q$) followed by $k \mapsto -k/\lambda$. The precise functional equation (1.7) follows provided we can show that

$$C_{p,q}(\lambda) = \lambda^{2pq-2} C_{q,p}(1/\lambda)$$

which indeed readily follows from the definition of $C_{p,q}(\lambda)$ in (2.4).

3. EXPANDING $f(k;\beta)$ IN TERMS OF DOTSENKO–FATEEV TYPE INTEGRALS

Here we will develop a strategy based on the integral formula (2.9) to expand $f(k,\beta)$ at order k^2. This relies on our ability to compute certain generalizations of a limiting case of the Dotsenko–Fateev integral. This same method has been used in refs. 8, 6 to compute the equivalent of $f(k,\beta)$ and its derivative at $k=0$.

We first expand the integrand in (2.9) as a function of k. According to (2.7) we have

$$\hat{F}(q,p,\lambda \mid \{x_i, y_j\}; k) = G(q,p,\lambda \mid \{x_i,y_j\}) \left(1 + \sum_{\nu=1}^{\infty} H_\nu(q,p,\lambda \mid \{x_i,y_j\})\, k^\nu\right)$$

where

$$\begin{aligned} G(q,p,\lambda \mid \{x_i,y_j\}) &= \frac{\prod_{i<i'} |x_i - x_{i'}|^{2\lambda} \prod_{j<j'} |y_j - y_{j'}|^{2/\lambda}}{\prod_{i=1}^{q} \prod_{j=1}^{p} (x_i+\lambda y_j)^2 \prod_{i=1}^{q} x_i^{1-\lambda} \prod_{j=1}^{p} y_j^{1-1/\lambda}} \\ 1+\sum_{\nu=1}^{\infty} H_\nu(q,p,\lambda \mid \{x_i,y_j\})\, k^\nu &= \frac{1}{\prod_{i=1}^{q} (1+kx_i/\lambda)^{1-\lambda} \prod_{j=1}^{p} (1-ky_j)^{1-1/\lambda}} \end{aligned} \tag{3.1}$$

The coefficients H_ν are homogeneous polynomials in $\{x_i, y_j\}$ of degree ν.

Let us now introduce the notation

$$I_{p,q,\lambda}[h(\{x_i,y_j\})] := \prod_{i=1}^{q} \int_0^\infty dx_i \prod_{j=1}^{p} \int_0^\infty dy_j\, Q_{p,q}^2 G(q,p,\lambda \mid \{x_i,y_j\}) \times \delta(1-Q_{p,q})\, h(\{x_i,y_j\}) \tag{3.2}$$

Because of the presence of the delta function the value of $I_{p,q,\lambda}$ is unchanged if $Q_{p,q}^2$ is replaced by $Q_{p,q}^n$ for any n. Doing this and also introducing the usual integral representation of the delta function, we see by a change of variables as detailed in ref. 6 that for h homogeneous of degree ν

$$I_{p,q,\lambda}[h(\{x_i,y_j\})] = \frac{J_{p,q,\lambda,n}[h(\{x_i,y_j\})]}{(\nu+n-1)!} = \frac{J_{p,q,\lambda}[h(\{x_i,y_j\})]}{(\nu-1)!} \tag{3.3}$$

where

$$J_{p,q,\lambda,n}[h(\{x_i, y_j\})]$$
$$:= \prod_{i=1}^{q} \int_0^\infty dx_i \prod_{j=1}^{p} \int_0^\infty dy_j\, Q_{p,q}^n G(q, p, \lambda \mid \{x_i, y_j\})\, e^{-Q_{p,q}} h(\{x_i, y_j\})$$

and $J_{p,q,\lambda} := J_{p,q,\lambda,0}$.

Recalling (2.9) and (3.1) we see that in terms of the notation (3.2)

$$f(k; \beta) = C_{p,q}(\lambda) \left(I_{p,q,\lambda}[1] + \sum_{\nu=1}^{\infty} I_{p,q,\lambda}[H_\nu(q, p, \lambda \mid \{x_i, y_j\})]\, k^\nu \right) \tag{3.4}$$

The definition of H_ν in (3.1) shows

$$H_2(q, p, \lambda \mid \{x_i, y_j\}) = \frac{(\lambda-1)^2}{2\lambda^2} \frac{Q_{p,q}^2}{(2\pi\rho)^2} - \frac{\lambda-1}{2\lambda^2} \left(\sum_{i=1}^{q} x_i^2 - \lambda \sum_{j=1}^{p} y_j^2 \right) \tag{3.5}$$

so to compute $f(k, \beta)$ at order k^2 our task is to evaluate

$$I_{p,q,\lambda}[Q_{p,q}^2] \qquad \text{and} \qquad I_{p,q,\lambda}\left[\sum_{i=1}^{q} x_i^2 - \lambda \sum_{j=1}^{p} y_j^2 \right] \tag{3.6}$$

Now because of the delta function in (3.2)

$$I_{p,q,\lambda}[Q_{p,q}^2] = I_{p,q,\lambda}[1] \tag{3.7}$$

and we know from ref. 6 that

$$C_{p,q}(\lambda)\, I_{p,q,\lambda}[1] = 1 \tag{3.8}$$

Thus our remaining task is to compute the second expression in (3.6) or equivalently, using (3.3), to compute

$$J_{p,q,\lambda}\left[\sum_{i=1}^{q} x_i^2 - \lambda \sum_{j=1}^{p} y_j^2 \right] = qJ_{p,q,\lambda}[x_i^2] - \lambda p J_{p,q,\lambda}[y_j^2] \tag{3.9}$$

where the second equality, valid for any $1 \leqslant i \leqslant q$ and $1 \leqslant j \leqslant p$, follows from the symmetry of the integrand.

For this purpose we first note formulas for $J_{p,q,\lambda}[h]$ in the cases $h=x_i^2$ and $h=y_j^2$. The formulas are

$$J_{p,q,\lambda}[x_i^2]=\frac{(2p-\lambda+1)}{2\pi\rho}J_{p,q,\lambda}[x_i]-\frac{p}{\pi\rho}J_{p,q,\lambda}\left[\frac{x_i^2}{x_i+\lambda y_j}\right] \tag{3.10}$$

$$J_{p,q,\lambda}[y_j^2]=\frac{(2q-1/\lambda+1)}{2\pi\rho}J_{p,q,\lambda}[y_j]-\frac{\lambda q}{\pi\rho}J_{p,q,\lambda}\left[\frac{x_j^2}{x_i+\lambda y_j}\right] \tag{3.11}$$

The derivation of (3.10) and (3.11) uses a technique based on the fundamental theorem of calculus. It was first used by Aomoto[1] in the context of the Selberg integral, and has been adapted in ref. 8 to the case of the Dotsenko–Fateev integral.

Let us give the details of the derivation of (3.10) (the derivation of (3.11) is similar). From the definition (3.1) we see that

$$\begin{aligned}
&\frac{\partial}{\partial x_i}G(q,p,\lambda\mid\{x_i,y_j\})\\
&\quad=\left(\frac{\lambda-1}{x_i}-2\sum_{j=1}^{p}\frac{1}{x_i+\lambda y_j}+2\lambda\sum_{i'=1;\,i'\neq i}^{q}\frac{1}{x_i-x_{i'}}\right)G(q,p,\lambda\mid\{x_i,y_j\})
\end{aligned}$$

Thus

$$\begin{aligned}
0&=\prod_{i=1}^{q}\int_0^\infty dx_i\prod_{j=1}^{p}\int_0^\infty dy_j\,\frac{\partial}{\partial x_i}\left(x_i^2G(q,p,\lambda\mid\{x_i,y_j\})\,e^{-\mathcal{Q}_{p,q}}\right)\\
&=(\lambda+1)\,J_{p,q,\lambda}[x_i]-2\sum_{j=1}^{p}J_{p,q,\lambda}\left[\frac{x_i^2}{x_i+\lambda y_j}\right]\\
&\quad+2\lambda\sum_{i'=1;\,i'\neq i}^{q}J_{p,q,\lambda}\left[\frac{x_i^2}{x_i-x_{i'}}\right]-2\pi\rho J_{p,q,\lambda}[x_i^2]\\
&=(\lambda+1)\,J_{p,q,\lambda}[x_i]-2pJ_{p,q,\lambda}\left[\frac{x_i^2}{x_i+\lambda y_j}\right]\\
&\quad+2\lambda(q-1)\,J_{p,q,\lambda}\left[\frac{x_i^2}{x_i-x_{i'}}\right]-2\pi\rho J_{p,q,\lambda}[x_i^2]
\end{aligned} \tag{3.12}$$

where the first equality follows from the fundamental theorem of calculus, while the final equality, valid for any $j=1,\ldots,p$ and any $i'=1,\ldots,q$, $(i'\neq i)$

follows by the symmetry of the integrand with respect to $\{x_i\}$ and $\{y_j\}$. The symmetry of the integrand with respect to $\{x_i\}$ also gives

$$J_{p,q,\lambda}\left[\frac{x_i^2}{x_i-x_{i'}}\right]=J_{p,q,\lambda}\left[\frac{x_{i'}^2}{x_{i'}-x_i}\right]$$

so we have

$$J_{p,q,\lambda}\left[\frac{x_i^2}{x_i-x_{i'}}\right]=\frac{1}{2}\left(J_{p,q,\lambda}\left[\frac{x_i^2}{x_i-x_{i'}}\right]+J_{p,q,\lambda}\left[\frac{x_{i'}^2}{x_{i'}-x_i}\right]\right)=J_{p,q,\lambda}[x_i]$$

Substituting in (3.12) implies (3.10).

From (3.10) and (3.11) we see that

$$\begin{aligned}
qJ_{p,q,\lambda}&[x_i^2]-\lambda pJ_{p,q,\lambda}[y_j^2]\\
&=\frac{q(2p-\lambda+1)}{2\pi\rho}J_{p,q,\lambda}[x_i]-\frac{\lambda p(2q-1/\lambda+1)}{2\pi\rho}J_{p,q,\lambda}[y_j]\\
&\quad-\frac{pq}{\pi\rho}J_{p,q,\lambda}\left[\frac{x_i^2-\lambda^2y_j^2}{x_i+\lambda y_j}\right]\\
&=\frac{q(-\lambda+1)}{2\pi\rho}J_{p,q,\lambda}[x_i]-\frac{\lambda p(-1/\lambda+1)}{2\pi\rho}J_{p,q,\lambda}[y_j]\\
&=-\frac{\lambda-1}{(2\pi\rho)^2}J_{p,q,\lambda}[Q_{p,q}]=-\frac{\lambda-1}{(2\pi\rho)^2}J_{p,q,\lambda}[1]
\end{aligned}\tag{3.13}$$

Recalling (3.5), the results (3.3), (3.7), (3.9) and (3.13) give that

$$I_{p,q,\lambda}[H_2(q,p,\lambda\mid\{x_i\},\{y_j\}]=\frac{1}{(2\pi\rho)^2}\frac{(\lambda-1)^2}{\lambda^2}J_{p,q,\lambda}[1]$$

Use of (3.8) then gives that the term proportional to k^2 in (3.4) is equal to

$$(\lambda-1)^2\left(\frac{k}{2\pi\lambda\rho}\right)^2=(\beta/2-1)^2\left(\frac{k}{\pi\beta\rho}\right)^2\tag{3.14}$$

It follows from this that the structure (1.8) is valid at order k^2 with

$$p_2(x)=(x-1)^2\tag{3.15}$$

4. LARGE-x EXPANSION OF $\rho^T_{(2)}(0, x)$

We have already remarked that the large-x expansion (1.3) of the charge–charge correlation, or what is the same thing for the one-component log–gas, the large-x expansion of $\rho^T_{(2)}(0, x)$, is equivalent to the small-k behaviour (1.4) of $S(k; \beta)$. More generally, as shown below, the non-oscillating part of this expansion is of the form

$$\rho^T_{(2)}(0, x)_{\text{non-osc}} \underset{x \to \infty}{\simeq} \sum_{n=1}^{\infty} \frac{c_n}{x^{2n}} \tag{4.1}$$

(with only even inverse powers of x), equivalent to the expansion

$$S_{\text{odd}}(k; \beta) \underset{k \to 0}{\simeq} \pi \sum_{n=1}^{\infty} \frac{(-1)^n c_n}{(2n-1)!} |k|^{2n-1} \tag{4.2}$$

where S_{odd} is that part of the expansion of $S(k, \beta)$ containing the terms singular in k (i.e., of odd order in $|k|$). This follows using the Fourier transform

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{x^{2n}} \, dx = \pi \frac{(-1)^n |k|^{2n-1}}{(2n-1)!}$$

from the theory of generalized functions (see e.g., ref. 8).

From the equivalence between (4.1) and (4.2) we see the fact, following from (3.14), that the term proportional to $|k|^3$ in the small k expansion of $S(k; \beta)$ is equal to

$$\rho(\beta/2-1)^2 \left(\frac{|k|}{\pi \beta \rho} \right)^3$$

is equivalent to the statement that the term proportional to $1/x^4$ in the non-oscillating part of the large x expansion of $\rho^T_{(2)}(0, x)$ is equal to

$$\rho^2 6\beta(\beta/2-1)^2 \left(\frac{1}{\pi \beta \rho x} \right)^4 \tag{4.3}$$

In this section we will derive (4.3) directly. We will also calculate the $O(1/x^6)$ term and so explicitly determine the $O(|k|^5)$ term in (4.2).

The starting point for our calculation is an exact β-dimensional integral formula for the two-particle distribution $\rho_{(2)}(0, x)$ valid for β even. With

$$S_n(a, b, c) := \prod_{j=0}^{n-1} \frac{\Gamma(a+1+jc)\, \Gamma(b+1+jc)\, \Gamma(1+(j+1)\, c)}{\Gamma(a+b+2+(N+j-1)\, c)\, \Gamma(1+c)}$$

the formula gives that in the thermodynamic limit[3]

$$\rho_{(2)}(0, x) = \rho^2(\beta/2)^\beta \frac{((\beta/2)!)^3}{\beta!\,(3\beta/2)!} \frac{e^{-\pi i\beta\rho x}(2\pi\rho x)^\beta}{S_\beta(1-2/\beta, 1-2/\beta, 2/\beta)} \int_{[0,1]} du_1 \cdots du_\beta$$
$$\times \prod_{j=1}^{\beta} e^{2\pi i\rho x u_j} u_j^{-1+2/\beta}(1-u_j)^{-1+2/\beta} \prod_{j<k} |u_k - u_j|^{4/\beta} \tag{4.4}$$

In a previous analysis[3] it has been shown that the non-oscillatory large-x behaviour is determined by the integrand in the vicinity of the endpoints 0 and 1, with the requirement that $\beta/2$ of the integration variables are in the vicinity of the endpoint 0, while the remaining $\beta/2$ integration variables are in the vicinity of the endpoint 1. Thus we write $u_{\beta/2+j} = 1 - v_j$ $(j = 1,..., \beta/2)$ (this introduces a combinatorial factor β choose $\beta/2$ to account for the different ways of so partitioning the integration variables) and then expand the integrand (excluding the exponential factors which involve x) in terms of the "small" variables u_j, v_j $(j = 1,..., \beta/2)$. In particular we must expand

$$\prod_{j=1}^{\beta/2} (1-u_j)^{-1+2/\beta}\,(1-v_j)^{-1+2/\beta} \prod_{l, l'=1}^{\beta/2} (1-u_l-v_{l'})^{4/\beta} \tag{4.5}$$

The function (4.5) is a symmetric function of the variables $\{u_j\}$ and $\{v_j\}$ separately. Let $\{q_\kappa\}_\kappa$ be a polynomial basis for symmetric functions with κ denoting a partition (ordered set of nonnegative integers) of no more than $\beta/2$ parts, and suppose furthermore that q_κ is homogeneous of order $|\kappa| := \kappa_1 + \cdots + \kappa_{\beta/2}$. Then we can write

$$\prod_{j=1}^{\beta/2} (1-u_j)^{-1+2/\beta}(1-v_j)^{-1+2/\beta} \prod_{l, l'=1}^{\beta/2} (1-u_l-v_{l'})^{4/\beta}$$
$$= \sum_{\kappa, \mu} w_{\kappa, \mu} q_\kappa(u_1,..., u_{\beta/2})\, q_\mu(v_1,..., v_{\beta/2}) \tag{4.6}$$

Substituting (4.6) in (4.4), then following the procedure of ref. 3, which involves extending the range of integration to $u_j \in (0, \infty)$, $v_j \in (0, \infty)$ and changing variables $u_j \mapsto 2\pi i\rho x u_j$, $v_j \mapsto -2\pi i\rho x v_j$ making use in the process of the fact that q_κ is homogeneous of degree $|\kappa|$, we obtain the non-oscillatory terms in the large-x asymptotic expansion of $\rho_{(2)}(0, x)$. This reads

$$\rho_{(2)}(0, x) \sim \rho^2 \binom{\beta}{\beta/2} (\beta/2)^\beta \frac{((\beta/2)!)^3}{\beta!\,(3\beta/2)!} \frac{1}{S_\beta(1-2/\beta, 1-2/\beta, 2/\beta)}$$
$$\times \sum_{\kappa, \mu} w_{\kappa, \mu} \frac{K_{\beta, \kappa} K_{\beta, \mu}}{i^{|\lambda| - |\mu|}(2\pi\rho x)^{|\kappa| + |\mu|}} \tag{4.7}$$

where

$$K_{\beta,\kappa} := \int_{[0,\infty)^{\beta/2}} du_1 \cdots du_{\beta/2} \prod_{l=1}^{\beta/2} u_l^{-1+2/\beta} e^{-u_l} \prod_{j<k} |u_k - u_j|^{4/\beta} q_\kappa(u_1,..., u_{\beta/2}) \tag{4.8}$$

The symmetry $w_{\kappa,\mu} = w_{\mu,\kappa}$ evident from (4.6) implies terms in (4.7) with $|\kappa| + |\mu|$ odd cancel. Therefore the sum in (4.7) can be restricted to partitions such that $|\kappa| + |\mu|$ is even, which means the asymptotic expansion only contains inverse even powers of x.

To proceed further we must be able to compute the expansion coefficients $w_{\kappa,\mu}$ as well as the integrals $K_{\beta,\kappa}$. For the former task it is convenient to choose q_κ equal to the monomial symmetric polynomial m_κ, which is defined as the symmetrization of the monomial $x_1^{\kappa_1} \cdots x_{\beta/2}^{\kappa_{\beta/2}}$ normalized so that the coefficient of $x_1^{\kappa_1} \cdots x_{\beta/2}^{\kappa_{\beta/2}}$ is unity.

First, we have the well known expansion

$$\prod_{j=1}^{n} (1-u_j)^a = \sum_{\ell(\kappa) \leqslant n} a_\kappa m_\kappa(u_1,..., u_n) \tag{4.9}$$

where

$$a_\kappa = \prod_{p=1}^{\ell(\kappa)} a_{\kappa_p}, \qquad a_k := \frac{(-a)_k}{k!} \tag{4.10}$$

with $\ell(\kappa)$ denoting the length of κ (i.e., number of non-zero parts). We can therefore immediately expand the first product in (4.6) in terms of monomial symmetric polynomials.

Consider next the expansion of the double product in (4.6). Making use of the formulas

$$(1-x)^a = \sum_{n=0}^{\infty} \frac{(-a)_n}{n!} x^n \tag{4.11}$$

$$\prod_{j=1}^{n} \left(\sum_{k=0}^{\infty} a_k t_j^k \right) = \sum_{\ell(\kappa) \leqslant n} a_0^{N-\ell(\kappa)} a_\kappa m_\kappa(\{t_j\}) \tag{4.12}$$

where a_κ is specified by the first equality in (4.10), we see that

$$\prod_{j=1}^{\beta/2} (1-u_j-v)^{4/\beta} = \sum_{\ell(\kappa) \leqslant \beta/2} (1-v)^{2-|\kappa|} c_\kappa m_\kappa(u_1,..., u_{\beta/2}) \tag{4.13}$$

where

$$c_\kappa = \prod_{p=1}^{\ell(\kappa)} c_{\kappa_p}, \qquad c_k := \frac{(-4/\beta)_k}{k!}$$

Expanding the factor $(1-v)^{2-|\kappa|}$ we can rewrite (4.13) as

$$\prod_{j=1}^{\beta/2} (1-u_j-v)^{4/\beta} = \sum_{n=0}^{\infty} w_n(u_1,..., u_{\beta/2}; \beta)\, v^n$$

for appropriate symmetric functions w_n. Replacing v by $v_{j'}$ and forming the product over j' using (4.12) we obtain

$$\prod_{j,\, j'=1}^{\beta/2} (1-u_j-v_{j'})^{4/\beta} = \sum_{\ell(\kappa) \leqslant \beta/2} w_0^{\beta/2-\ell(\kappa)} w_\kappa m_\kappa(v_1,..., v_{\beta/2})$$

where $w_\kappa := \prod_{p=1}^{\ell(\kappa)} w_{\kappa_p}$. The final step is to expand $w_0^{\beta/2-\ell(\kappa)} w_\kappa$, in terms of $\{m_\mu\}$ and so obtain the expansion

$$\prod_{j,\, j'=1}^{\beta/2} (1-u_j-v_{j'})^{4/\beta} = \sum_{\mu,\,\kappa} t_{\mu,\,\kappa} m_\mu(u_1,..., u_{\beta/2})\, m_\kappa(v_1,..., v_{\beta/2}) \tag{4.14}$$

The practical implementation of this procedure requires the use of computer algebra. We work with arbitrary (positive integer) values of $\beta/2$. Furthermore, we only include terms with $|\mu|+|\kappa| \leqslant 6$ throughout since according to (4.7) these terms suffice for the evaluation of the coefficients of $1/x^{2n}$, $n \leqslant 3$.

Having obtained the coefficients $t_{\mu,\,\kappa}$, in (4.14), we multiply the series (4.14) with the two series of the form (4.9) representing the first two products in (4.5), expressing the answer in the form of (4.6), and so determining the coefficients $w_{\kappa,\,\mu}$. Again this step requires computer algebra.

With $w_{\kappa,\,\mu}$ in (4.6) determined, it remains to compute the multiple integral (4.8) with $q_\mu = m_\mu$. For this task we introduce a further basis of symmetric functions, namely the Jack polynomials $\{P_\kappa^{(\beta/2)}(u_1,..., u_{\beta/2})\}$. The Jack polynomials $P_\kappa^{(2/\beta)}(z_1,..., z_N)$ with $z_j := e^{2\pi i x_j/L}$, when multiplied by the ground state wave function $|0\rangle$, are the eigenfunctions of the Calogero–Sutherland Schrödinger operator (2.1).[5] Each polynomial is homogeneous of degree $|\kappa|$ and has the expansion

$$P_\kappa^{(\alpha)}(z_1,..., z_N) = m_\kappa + \sum_{\mu<\kappa} a_{\kappa\mu} m_\mu \tag{4.15}$$

where $<$ is the dominance partial ordering for partitions: $\mu < \kappa$ if $|\kappa| = |\mu|$ with $\kappa \neq \mu$ and $\sum_{i=1}^{p} \mu_i \leqslant \sum_{i=1}^{p} \kappa_i$ for each $p = 1,..., N$. The coefficients $a_{\kappa\mu}$ can be calculated by recurrence.[19]

The significance of the Jack polynomial basis is that we have the explicit integral evaluation

$$\frac{1}{W_{a\alpha N}} \prod_{l=1}^{N} \int_0^\infty dt_l\, t_l^a e^{-t_l} P_\kappa^{(\alpha)}(t_1,..., t_N) \prod_{j<k} |t_k - t_j|^{2/\alpha}$$
$$= P_\kappa^{(\alpha)}(1^N)[a + (N-1)/\alpha + 1]_\kappa^{(\alpha)} \tag{4.16}$$

which is a limiting case of an integration formula due to Macdonald,[19] Kadell,[15] and Kaneko.[17] In (4.16)

$$W_{a\alpha N} = \prod_{l=1}^{N} \int_0^\infty dt_l\, t_l^a e^{-t_l} \prod_{j<k} |t_k - t_j|^{2/\alpha}$$
$$= \prod_{j=0}^{N-1} \frac{\Gamma(1 + (j+1)/\alpha)\, \Gamma(a + 1 + j/\alpha)}{\Gamma(1 + 1/\alpha)}$$
$$[u]_\kappa^{(\alpha)} := \prod_{j=1}^{N} \frac{\Gamma(u - (j-1)/\alpha + \kappa_j)}{\Gamma(u - (j-1)/\alpha)}$$

and $P_\kappa^{(\alpha)}(1^N)$ denotes $P_\kappa^{(\alpha)}(x_1,..., x_N)$ evaluated at $x_1 = \cdots = x_N = 1$.

To make use of (4.16) we must first express the monomial symmetric polynomials m_κ in terms of $\{P_\mu^{(2/\beta)}\}_{\mu \leqslant \kappa}$, which can be done using computer algebra from knowledge of the expansion (4.15). Substituting in (4.16) allows the integrals $K_{\kappa\beta}$ to be computed.

After completing this procedure all terms in (4.7) for $|\kappa| + |\mu| \leqslant 6$ are known explicitly. Performing the sum and simplifying we obtain

$$\rho_{(2)}(0, x) \sim \rho^2 \left(1 - \frac{1}{\beta(\pi\rho x)^2} + \frac{3(\beta-2)^2}{2\beta^3(\pi\rho x)^4} - \frac{15(\beta-2)^2\,(\beta^2 - 3\beta + 4)}{2\beta^5(\pi\rho x)^6} + \cdots\right) \tag{4.17}$$

Note that this agrees with the known form (1.3) for the term $O(1/x^2)$, and the form (4.3) for the term $O(1/x^4)$. The term $O(1/x^6)$, due to the equivalence between (4.1) and (4.2), implies the term $O(|k|^5)$ in the small-k expansion of $S(k, \beta)$ is equal to

$$(\beta/2 - 1)^2 \left((\beta/2)^2 - \frac{3}{2}(\beta/2) + 1\right)\left(\frac{|k|}{\pi\beta}\right)^5 \tag{4.18}$$

This is of the form of the conjecture (1.8) with

$$p_4(x) = (x-1)^2\,(x^2 - \tfrac{3}{2}x + 1) \tag{4.19}$$

5. $S(k; \beta)$ FOR SPECIAL β

Let us assume the validity of (1.8). The coefficients specifying the polynomials $p_j(x)$ therein can be determined from knowledge of the coefficient of $|k|^{j+1}$ in $S(k;\beta)$ or $\partial^p S(k;\beta)/\partial\beta^p$ $(p \leqslant j)$ at special values of β. Now in the context of random matrix theory $S(k;\beta)$ has been evaluated in terms of elementary functions for $\beta = 1$, 2 and 4. The results are[20]

$$S(k;1) = \begin{cases} \dfrac{|k|}{\pi} - \dfrac{|k|}{2\pi}\log\left(1 + \dfrac{|k|}{\pi\rho}\right), & |k| \leqslant 2\pi\rho \\ 2\rho - \dfrac{|k|}{2\pi}\log\left(\dfrac{1+|k|/\pi\rho}{-1+|k|/\pi\rho}\right) & |k| \geqslant 2\pi\rho \end{cases} \tag{5.1}$$

$$S(k;2) = \begin{cases} \dfrac{|k|}{2\pi} & |k| \leqslant 2\pi\rho \\ \rho & |k| \geqslant 2\pi\rho \end{cases} \tag{5.2}$$

$$S(k;4) = \begin{cases} \dfrac{|k|}{4\pi} - \dfrac{|k|}{8\pi}\log\left|1 - \dfrac{|k|}{2\pi\rho}\right| & |k| \leqslant 4\pi\rho \\ \rho & |k| \geqslant 4\pi\rho \end{cases} \tag{5.3}$$

Recalling the definition (1.6) of $f(k,\beta)$ we read off

$$f(k;1) = 1 - \frac{1}{2}\log\left(1 + \frac{k}{\pi\rho}\right) \tag{5.4}$$

$$f(k;2) = 1 \tag{5.5}$$

$$f(k;4) = 1 - \frac{1}{2}\log\left(1 - \frac{k}{2\pi\rho}\right) \tag{5.6}$$

The exact evaluation (5.5) implies that for all j $p_j(x)$ contains a factor of $(x-1)$. In the case of j odd this gives no new information since the factor $(x-1)$ was already deduced as a consequence of the functional equation (1.9). On the other hand, in the case j even this fact together with the functional equation (1.9) implies

$$p_j(x) = (x-1)^2 \sum_{l=0}^{j-2} b_{j,l}\,x^l, \qquad b_{j,l} = b_{j,\,j-2-l} \qquad (j \text{ even}) \tag{5.7}$$

Consider now the constraints on the coefficients in (5.7) and (1.11) which follow from (5.4) and (5.6). As (5.4) and (5.6) are related by the functional equation (1.7), and this is built into the structures (5.7) and (1.11), only one of these exact evaluations gives distinct information on $p_j(x)$. For definiteness consider (5.4). We see that

$$[k^j]\, f(k;1) = \frac{1}{2} \frac{(-1)^j}{j(\pi\rho)^j}, \qquad j \geqslant 1 \tag{5.8}$$

where the notation $[k^j]$ denotes the coefficient of k^j. Recalling (1.8), (5.7) and (1.11) this implies, for j even,

$$\begin{aligned}\frac{1}{j} = \frac{1}{2}\big(&(1+2^{-(j-2)})\, b_{j,0} + (2^{-1} + 2^{-(j-3)})\, b_{j,1} + \cdots \\ &+ (2^{-j/2+2} + 2^{-j/2})\, b_{j,j/2-2} + 2^{-j/2+1} b_{j,j/2-1}\big)\end{aligned} \tag{5.9}$$

while for j odd

$$\begin{aligned}\frac{1}{j} = \big(&(1+2^{-(j-1)})\, \tilde{a}_{j,0} + (2^{-1} + 2^{-(j-2)})\, \tilde{a}_{j,1} + \cdots \\ &+ (2^{-(j-1)/2+1} + 2^{-(j-1)/2-1})\, \tilde{a}_{j,(j-1)/2-1} \\ &+ 2^{-(j-1)/2} \tilde{a}_{j,(j-1)/2}\big)\end{aligned} \tag{5.10}$$

In the case $j=1$ (5.10) gives $\tilde{a}_{j,0}=1$ which reclaims (1.12), while in the case $j=2$ (5.9) gives $b_{j,0}=1$ which reclaims (3.15).

The exact form of $S(k;\beta)$ in the weak coupling scaling limit $\beta \to 0$, $k \to 0$, k/β fixed is also available. Introducing the dimensionless Fourier transforms

$$\tilde{S}(k;\beta) := \rho \int_{-\infty}^{\infty} (\rho^T_{(2)}(0,x) + \rho\delta(x))\, e^{i\rho x k}\, dx$$

$$\tilde{\Phi}(k) := \rho \int_{-\infty}^{\infty} \Phi(x)\, e^{i\rho x k}\, dx$$

where $\Phi(x) := -\log|x|$ is the pair potential of the log–gas (thus the integral in the definition of the $\tilde{\Phi}(k)$ is to be interpreted as a generalized function) we have[12]

$$\tilde{S}(k;\beta) \sim 1 - \frac{\beta\tilde{\Phi}(k)}{1+\beta\tilde{\Phi}(k)} \tag{5.11}$$

Since

$$\tilde{\Phi}(k) = \frac{\pi}{|k|} \tag{5.12}$$

and noting $\tilde{S}(k;\beta) = S(k\rho;\beta)/\rho$ we thus have that in the weak coupling scaling limit

$$S(k,\beta) = \rho\left(1 - \frac{1}{1+|k|/\pi\beta\rho}\right) \tag{5.13}$$

Expanding (5.13) in the form (1.8) and recalling (5.7) and (1.11) we deduce

$$\tilde{a}_{j,0} = 1 \qquad \text{and} \qquad b_{j,0} = 1 \tag{5.14}$$

for all j. Using (5.14) in (5.9) and (5.10) gives that in the case $j=3$, $\tilde{a}_{j,1} = -\frac{11}{6}$, and in the case $j=4$, $b_{j,1} = -\frac{3}{2}$. The latter result reclaims (4.18) while the former result together with (5.14) gives

$$p_3(x) = (x-1)(1 - \tfrac{11}{6}x + x^2) \tag{5.15}$$

An alternative way to derive (5.14) is to consider the $\beta \to \infty$ low temperature limit. In this limit the system behaves like an harmonic crystal, for which we have available the analytic formula[(4)] [4]

$$\rho_{(2)}^{(\text{har})}(x;0) = \rho^2 \sum_{p=-\infty;\, p\neq 0}^{\infty} \left(\frac{\beta}{4\pi f(p)}\right)^{1/2} e^{-\beta(p-\rho x)^2/4f(p)} \tag{5.16}$$

where

$$f(p) = \frac{1}{\pi^2}\int_0^{1/2} \frac{1-\cos 2\pi p t}{t-t^2}\, dt$$

Taking the Fourier transform gives for $|k| < 2\pi\rho$

$$\begin{aligned} S^{(\text{har})}(k;\beta) &= \rho \sum_{p=-\infty}^{\infty} (e^{-k^2 f(p)/\beta\rho^2} - 1)\, e^{ikp/\rho} \\ &\underset{\beta\to\infty}{\simeq} -\rho\, \frac{k^2}{\beta\rho^2} \sum_{p=-\infty}^{\infty} f(p)\, e^{ikp/\rho} = \frac{|k|/\pi\beta}{1-|k|/2\pi\rho} \end{aligned} \tag{5.17}$$

[4] The denominator of the exponent in (3.10) of ref. 4 contains a spurious factor of π^2 which is corrected in (5.16).

This formula maps to the weak coupling result (5.11) under the action of the functional equation (1.7) and so implies (5.14).

6. PERTURBATION ABOUT $\beta = 0$

The formula (5.13) is just the first term in a systematic weak coupling renormalized Mayer series expansion in β. In the case of the two-dimensional one-component plasma, low order terms of this expansion have recently been analyzed by Kalinay *et al.*[16] Results from that study can readily be transcribed to the case of the one-component log–gas.

Formally, the renormalized Mayer series expansion is for the dimensionless free energy $\beta \bar{F}^{\text{ex}}$ (in ref. 16 our $\beta \bar{F}^{\text{ex}}$ is written $-\beta \bar{F}^{\text{ex}}$), and one computes the direct correlation function via the functional differentiation formula

$$c(0, x) = -\frac{\delta^2(\beta \bar{F}^{\text{ex}})}{\delta \rho_{(1)}(0)\, \delta \rho_{(1)}(x)} \tag{6.1}$$

The Ornstein–Zernicke relation gives that the dimensionless Fourier transform of the direct correlation function, $\tilde{c}(k, \beta)$ say, is related to the dimensionless structure function $\tilde{S}(k; \beta)$ by

$$\tilde{c}(k; \beta) = 1 - \frac{1}{\tilde{S}(k; \beta)} \tag{6.2}$$

so expanding $\tilde{c}(k, \beta)$ about $\beta = 0$ with k/β fixed is equivalent to expanding $\tilde{S}(k; \beta)$ about $\beta = 0$ with k/β fixed.

Now, transcribing the results of ref. 16 we read off that the weak coupling diagrammatic expansion of $c(x_1, x_2)$ starts as

$$c(x_1, x_2) = -\beta \Phi(x_1, x_2) + \frac{1}{2!}(K(x_1, x_2))^2 + \cdots \tag{6.3}$$

where

$$K(x_1, x_2) = -\beta\pi \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(x_1 - x_2)}}{|k| + \kappa} \tag{6.4}$$

with $\kappa = \beta\pi\rho$. This implies

$$\tilde{c}(k; \beta) = -\frac{\beta\pi}{|k|} + \frac{1}{2}\rho \int_{-\infty}^{\infty} \frac{dl}{2\pi} \frac{\beta\pi}{|l| + \kappa} \frac{\beta\pi}{|\rho k - l| + \kappa} \tag{6.5}$$

The integral is straightforward (consider separately the ranges of l such that $l>0$ ($l<0$) and $\rho k-l>0$ ($\rho k-l<0$)). In terms of $k' := \rho k/\kappa = k/\pi\beta$,

$$\tilde{c}(k;\beta) = -\frac{1}{|k'|} + \beta\,\frac{1+|k'|}{|k'|(2+|k'|)}\log(1+|k'|) + O(\beta^2) \tag{6.6}$$

or equivalently using (6.2)

$$S(k;\beta) = \rho\,\frac{|k/\kappa|}{1+|k/\kappa|} + \beta\rho\,\frac{|k/\kappa|}{(1+|k/\kappa|)(2+|k/\kappa|)}\log(1+|k/\kappa|) + O(\beta^2) \tag{6.7}$$

Notice that the leading order term in (6.7) reproduces (5.11).

The exact result (6.7) gives the explicit value of the coefficient of x in the polynomial $p_j(x)$. Thus recalling (1.11) and (5.7) we have

$$\begin{aligned}\frac{1}{2}(b_{j,1}-2) &= [x^j]\,\frac{1}{(1+x)(2+x)}\log(1+x), \qquad (j \text{ even})\\ \frac{1}{2}(1-\tilde{a}_{j,1}) &= [x^j]\,\frac{1}{(1+x)(2+x)}\log(1+x), \qquad (j \text{ odd})\end{aligned} \tag{6.8}$$

Furthermore, a simple calculation gives

$$[x^j]\,\frac{1}{(1+x)(2+x)}\log(1+x) = (-1)^j \sum_{q=1}^{j} \frac{1}{q}(1-2^{q-j}) \tag{6.9}$$

so we have for example

$$\begin{gathered}\tilde{a}_{5,1} = -\tfrac{91}{30}, \qquad b_{6,1} = -\tfrac{31}{15}, \qquad \tilde{a}_{7,1} = -\tfrac{1607}{420}\\ b_{8,1} = -\tfrac{263}{84}, \qquad \tilde{a}_{9,1} = -\tfrac{791}{180}\end{gathered} \tag{6.10}$$

Substituting $\tilde{a}_{5,1}$ from (6.10) and $\tilde{a}_{5,0}$ from (5.14) in (5.9) shows $\tilde{a}_{5,2} = \frac{62}{15}$. Similarly, the value of $b_{6,1}$ above allows us to deduce that $b_{6,2} = \frac{13}{4}$. Thus we have

$$\begin{aligned}p_5(x) &= (x-1)(x^4 - \tfrac{91}{30}x^3 + \tfrac{62}{15}x^2 - \tfrac{91}{30}x + 1)\\ p_6(x) &= (x-1)^2\,(x^4 - \tfrac{37}{15}x^3 + \tfrac{13}{4}x^2 - \tfrac{37}{15}x + 1)\end{aligned} \tag{6.11}$$

We remark that according to the conjecture (1.8), the expansion of $S(k,\beta)$ about $\beta=0$ should have the structure

$$S(k,\beta) = f_0(k/\kappa) + \beta f_1(k/\kappa) + \beta^2(f_2(k/\kappa) + \cdots \tag{6.12}$$

where

$$f_j(u) = u^j(c_{j,0} + c_{j,1}u + \cdots) \tag{6.13}$$

Consideration of the analysis of ref. 16 reveals that the structure (6.12) will indeed result from the weak coupling expansion, however the structure (6.13) is not immediately evident. (Of course the explicit form f_2 as revealed by (6.7) exhibits this structure.)

7. PERTURBATION ABOUT β = 2 AND β = 4

A feature of the couplings $\beta = 1$, 2 and 4 is that the n-particle distribution functions are known for each $n = 2, 3,...$.[20] Introducing the dimensionless distribution

$$g(x_1,..., x_n) := \rho_{(n)}(x_1,..., x_n)/\rho^n$$

we can use our knowledge of $g(x_1,, x_n)$ for $n = 2$, 3 and 4 at these specific β to expand $g(x_1, x_2)$ about $\beta = \beta_0$ to first order in $\beta - \beta_0$. Thus with $\Phi(x_1, x_2) := -\log|x_1 - x_2|$ we have[13]

$$\begin{aligned} g(x_1, x_2; \beta) = g(x_1, x_2) + (\beta - \beta_0) \Big\{ &- g(x_1, x_2)\, \Phi(x_1, x_2) \\ &- 2\rho \int_{-\infty}^{\infty} (g(x_1, x_2, x_3) - g(x_1, x_2))\, \Phi(x_1, x_3)\, dx_3 \\ &- \tfrac{1}{2}\rho^2 \int_{-\infty}^{\infty} (g(x_1, x_2, x_3, x_4) - g(x_1, x_2)\, g(x_3, x_4) \\ &- g(x_1, x_2, x_3) - g(x_1, x_2, x_4) + 2g(x_1, x_2)) \\ &\times \Phi(x_3, x_4)\, dx_3\, dx_4 \Big\} + O((\beta - \beta_0)^2) \end{aligned} \tag{7.1}$$

where on the right hand side the dimensionless distributions are evaluated at $\beta = \beta_0$. Here we will compute this first order correction, and the corresponding first order correction for $S(k; \beta)$, in the cases $\beta_0 = 2$ and $\beta_0 = 4$ (we do not consider $\beta_0 = 1$ because of its relation to $\beta_0 = 4$ via the functional equation (1.7)).

Now, in the case $\beta_0 = 2$ we have

$$g(x_1,..., x_n) = \det[P_2(x_j, x_k)]_{j,k=1,...,n}, \qquad P_2(x, y) := \frac{\sin \pi\rho(x - y)}{\pi\rho(x - y)} \tag{7.2}$$

while in the case $\beta_0 = 4$

$$g(x_1,..., x_n) = \mathrm{qdet}[P_4(x_j, x_k)]_{j,k=1,...,n} \tag{7.3}$$

where

$$P_4(x_j, x_k) = \begin{bmatrix} \dfrac{\sin 2\pi\rho x_{jk}}{2\pi\rho x_{jk}} & \mathrm{Si}(2\pi\rho x_{jk}) \\ \dfrac{1}{2\pi\rho}\dfrac{d}{dx_{jk}}\left(\dfrac{\sin 2\pi\rho x_{jk}}{2\pi\rho x_{jk}}\right) & \dfrac{\sin 2\pi\rho x_{jk}}{2\pi\rho x_{jk}} \end{bmatrix} \tag{7.4}$$

with $x_{jk} := x_j - x_k$ and $\mathrm{Si}(x)$ denoting the complimentary sine integral, defined in terms of the sine integral $\mathrm{si}(x)$ by

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt = \frac{\pi}{2} + \mathrm{si}(x), \qquad \mathrm{si}(x) := -\int_x^\infty \frac{\sin t}{t}\,dt \tag{7.5}$$

In (7.3) qdet denotes quaternion determinant, which can be defined as

$$\begin{aligned} &\mathrm{qdet}[P_4(x_j, x_k)]_{j,k=1,...,n} \\ &\quad = \sum_{P \in S_n} (-1)^{n-l} \prod_1^l (P_4(x_a, x_b)\, P_4(x_b, x_c) \cdots P_4(x_d, x_a))^{(0)} \end{aligned} \tag{7.6}$$

where the superscript (0) denotes the operation $\frac{1}{2}\mathrm{Tr}$, P is any permutation of the indicies $(1,..., n)$ consisting of l exclusive cycles of the form $(a \to b \to c \cdots \to d \to a)$ and $(-1)^{n-l}$ is equal to the parity of P. Note that this reproduces the definition of an ordinary determinant in the case that P_4 is a multiple of the identity.

The task now is to substitute (7.2) in the case $\beta_0 = 2$ and (7.3) in the case $\beta_0 = 4$, and to compute the integrals. Consider first the case $\beta_0 = 2$. After some calculation (see Appendix A) we find

$$\begin{aligned} g(0, x; \beta) = 1 - \left(\frac{\sin \pi\rho x}{\pi\rho x}\right)^2 + (\beta - 2)\left\{\frac{1}{2}\left(\frac{\sin \pi\rho x}{\pi\rho x}\right)^2 - \frac{\sin 2\pi\rho x}{2\pi\rho x} + \mathrm{ci}(2\pi\rho x)\right. \\ \left. + \frac{1}{2(\pi\rho x)^2}((\log 2\pi\rho\,|x| + C)\cos 2\pi\rho x - \mathrm{ci}(2\pi\rho x))\right\} \\ + O((\beta - 2)^2) \end{aligned} \tag{7.7}$$

where C denotes Euler's constant while

$$\mathrm{ci}(x) = C + \log|x| + \int_0^x \frac{\cos t - 1}{t}\,dt = -\int_x^\infty \frac{\cos t}{t}\,dt \tag{7.8}$$

denotes the cosine integral. From this we can compute (again see Appendix A) that up to terms $O((\beta-2)^2)$

$$S(k;\beta) = \begin{cases} \dfrac{|k|}{2\pi} + (\beta-2)\,\rho\left\{\dfrac{1}{2}\log\left(1-\dfrac{k^2}{(2\pi\rho)^2}\right)\right. \\ \qquad\left. + \dfrac{|k|}{4\pi\rho}\log\dfrac{2\pi\rho+|k|}{2\pi\rho-|k|} - \dfrac{|k|}{4\pi\rho}\right\}, & |k|<2\pi\rho \\ \rho + (\beta-2)\,\rho\left\{\dfrac{1}{2}\log\dfrac{|k|+2\pi\rho}{|k|-2\pi\rho}\right. \\ \qquad\left. + \dfrac{|k|}{4\pi\rho}\log\left(1-\dfrac{(2\pi\rho)^2}{k^2}\right) - \dfrac{\pi\rho}{|k|}\right\}, & |k|>2\pi\rho \end{cases} \tag{7.9}$$

Let us consider the consequence of (7.9) in regards to the expansion (1.8). For $|k|<2\pi\rho$ we observe that all terms but the one proportional to $|k|$ are even in k. This is consistent with $p_j(x)$ having the quadratic factor $(x-1)^2$ for j odd (recall (5.7)), but only a linear factor for j even (recall (1.11)). Moreover, we can use (7.9) to derive a linear equation for the coefficients $\{\tilde{a}_j\}$. First we differentiate (7.9) with respect to β, set $\beta=2$ and expand about $k=0$ to obtain

$$\left.\frac{\partial S(k;\beta)}{\partial\beta}\right|_{\beta=2} = -\frac{1}{2}\frac{|k|}{2\pi\rho} + \sum_{j=1}^\infty \frac{1}{2j(2j-1)}\left(\frac{|k|}{2\pi\rho}\right)^{2j}, \qquad |k|<2\pi\rho$$

Recalling (1.8) and (5.7) this in turn implies

$$\frac{1}{2j(2j-1)} = \frac{1}{2}\left(2\tilde{a}_{2j-1,0} + 2\tilde{a}_{2j-1,1} + \cdots + 2\tilde{a}_{2j-1,j-2} + \tilde{a}_{2j-1,j-1}\right) \tag{7.10}$$

In the case $j=4$ we deduce from this equation, (5.14), (6.10) and (5.10) that

$$p_7(x) = (x-1)\left(1 - \tfrac{1607}{420}x + \tfrac{2011}{280}x^2 - \tfrac{911}{105}x^3 + \tfrac{2011}{280}x^4 - \tfrac{1607}{420}x^5 + x^6\right) \tag{7.11}$$

Consider now the case $\beta_0=4$. Due to P_4 in (7.3) being a 2×2 matrix, the calculation required to compute (7.1) is more lengthy and tedius than in the case $\beta_0=2$, although the common structure of n-point distributions means the two cases are analogous. Some details are given in Appendix B. Our final expression for $g(x_1, x_2; \beta)$ is given by (B.4). We find its Fourier transform can be computed explicitly in terms of elementary functions, together with the dilogarithm

$$\operatorname{dilog}(x) := \int_1^x \frac{\log t}{1-t}\, dt \tag{7.12}$$

Explicitly, with $\rho = 1$ for notational convenience, up to terms $O((\beta-4)^2)$

$$\begin{aligned} S(k,\beta) = S(k,4) + (\beta-4)\bigg(&-\frac{\pi}{|k|} + \hat{B}_0(k) + 2\hat{B}_1(k) \\ &- 4\hat{B}_3(k) + 2\hat{B}_5(k) + \hat{B}_6(k) - \hat{B}_7(k)\bigg) \end{aligned} \tag{7.13}$$

where

$$\begin{aligned} \hat{B}_0(k) = {} & -\frac{3}{2} + \frac{3\,|k|}{8\pi} + \frac{|k|}{4\pi} \log\left(\frac{4\pi+|k|}{|k|}\right) \\ & + \left(C + \frac{1}{2}\log(16\pi^2 - k^2)\right)\left(1 - \frac{|k|}{4\pi} + \frac{|k|}{8\pi}\log\left|1-\frac{|k|}{2\pi}\right|\right) \\ & + \frac{|k|}{16\pi}\left(\operatorname{dilog}\left(\frac{|k|}{2\pi+|k|}\right) - \operatorname{dilog}\left(\frac{4\pi+|k|}{2\pi+|k|}\right)\right. \\ & \left. - \log\left|1-\frac{|k|}{2\pi}\right| \log\left(\frac{4\pi+|k|}{|k|}\right) + g_1(k)\right) \\ & + \frac{2\pi-|k|}{8\pi}\log\left|1-\frac{|k|}{2\pi}\right|, \qquad |k| < 4\pi \end{aligned} \tag{7.14}$$

$$\begin{aligned} \hat{B}_0(k) = {} & \frac{1}{2}\log\left(\frac{|k|+4\pi}{|k|-4\pi}\right) + \frac{|k|}{8\pi}\log\left(\frac{k^2-16\pi^2}{k^2}\right) + \frac{|k|}{16\pi}\left(\operatorname{dilog}\left(\frac{|k|}{|k|+2\pi}\right)\right. \\ & + \operatorname{dilog}\left(\frac{|k|}{|k|-2\pi}\right) - \operatorname{dilog}\left(\frac{|k|+4\pi}{|k|+2\pi}\right) \\ & \left. - \operatorname{dilog}\left(\frac{|k|-4\pi}{|k|-2\pi}\right)\right), \qquad |k| > 4\pi \end{aligned} \tag{7.15}$$

$$\hat{B}_1(k) = \begin{cases} \dfrac{\pi}{|k|}\left(1 - \dfrac{|k|}{4\pi} + \dfrac{|k|}{8\pi}\log\left|1-\dfrac{|k|}{2\pi}\right|\right), & |k| < 4\pi \\ 0, & |k| > 4\pi \end{cases} \tag{7.16}$$

$$\begin{aligned}
\hat{B}_3(k) = &-\frac{3}{2}+\frac{3\,|k|}{8\pi}+C\left(1-\frac{|k|}{4\pi}+\frac{|k|}{8\pi}\log\left|1-\frac{|k|}{2\pi}\right|\right)\\
&+\left(\frac{1}{8}-\frac{3\,|k|}{32\pi}\right)\log\left|1-\frac{|k|}{2\pi}\right|\\
&+\frac{|k|}{64\pi}\left(\log\left|1-\frac{|k|}{2\pi}\right|\right)^2+\frac{1}{8\pi}(4\pi-|k|)\log(4\pi-|k|)\\
&-\frac{1}{8\pi}\,|k|\log|k|+\frac{1}{2}\log 4\pi\\
&+\frac{|k|}{32\pi}\left(\operatorname{dilog}\left(\frac{|k|}{2\pi+|k|}\right)+\frac{\pi^2}{12}-\operatorname{dilog}\left(\frac{4\pi}{2\pi+|k|}\right)\right.\\
&-\operatorname{dilog}\left(\frac{|k|}{2\pi}\right)-\operatorname{dilog}\left(\frac{4\pi-|k|}{2\pi}\right)+2\log(2\pi)\log\left|1-\frac{|k|}{2\pi}\right|\\
&\left.+\log(2\pi+|k|)\log\left|1-\frac{|k|}{2\pi}\right|+g_2(k)\right),\qquad |k|<4\pi
\end{aligned}$$

$$\hat{B}_3(k)=0,\qquad |k|>4\pi \tag{7.17}$$

$$\hat{B}_5(k)=\begin{cases}\hat{B}_3(k)-\dfrac{|k|}{128\pi}\left(\log\left|1-\dfrac{|k|}{2\pi}\right|\right)^2+\dfrac{|k|}{32\pi}\,g_3(k), & |k|<4\pi\\ 0, & |k|>4\pi\end{cases} \tag{7.18}$$

$$\begin{aligned}
\hat{B}_6(k) = &-\frac{3}{2}+\frac{3\,|k|}{8\pi}-\frac{|k|}{16\pi}\log\left|1-\frac{|k|}{2\pi}\right|\\
&+(C+\log(4\pi-|k|))\left(1-\frac{|k|}{4\pi}+\frac{|k|}{8\pi}\log\left|1-\frac{|k|}{2\pi}\right|\right)\\
&+\frac{|k|}{32\pi}\left(\frac{\pi^2}{3}-\operatorname{dilog}\left(\frac{|k|}{2\pi}\right)-\log\left|1-\frac{|k|}{2\pi}\right|\log\left(\frac{|k|}{2\pi}\right)\right.\\
&-2\operatorname{dilog}\left(\frac{4\pi-|k|}{2\pi}\right)\\
&\left.-2\log\left|1-\frac{|k|}{2\pi}\right|\log\left(\frac{4\pi-|k|}{2\pi}\right)+g_4(k)\right),\qquad |k|<4\pi
\end{aligned}$$

$$\hat{B}_6(k) = 0, \qquad |k| > 4\pi \tag{7.19}$$

$$\hat{B}_7(k) = \begin{cases} \dfrac{\pi}{|k|}\left(1 - \dfrac{|k|}{4\pi} + \dfrac{|k|}{8\pi}\log\left|1 - \dfrac{|k|}{2\pi}\right|\right)^2, & |k| < 4\pi \\ 0, & |k| > 4\pi \end{cases} \tag{7.20}$$

with

$$g_1(k) = \begin{cases} \operatorname{dilog}\left(\dfrac{4\pi - |k|}{2\pi - |k|}\right) - \operatorname{dilog}\left(\dfrac{2\pi}{2\pi - |k|}\right) \\ \qquad -\dfrac{\pi^2}{6} - \log\left(1 - \dfrac{|k|}{2\pi}\right)\log\left(\dfrac{4\pi - |k|}{2\pi - |k|}\right), & |k| < 2\pi \\ \operatorname{dilog}\left(\dfrac{|k|}{|k| - 2\pi}\right) - \operatorname{dilog}\left(\dfrac{2\pi}{|k| - 2\pi}\right) \\ \qquad -\dfrac{\pi^2}{6} + \log\left(\dfrac{2\pi}{|k| - 2\pi}\right)\log\left(\dfrac{|k|}{|k| - 2\pi}\right), & 2\pi < |k| < 4\pi \end{cases}$$

$$g_2(k) = \begin{cases} \operatorname{dilog}\left(\dfrac{4\pi - |k|}{2\pi - |k|}\right) - \dfrac{\pi^2}{6} \\ \qquad + \log(2\pi - |k|)\log\left(1 - \dfrac{|k|}{2\pi}\right), & |k| < 2\pi \\ -\operatorname{dilog}\left(\dfrac{2\pi}{|k| - 2\pi}\right) \\ \qquad + \log(4\pi - |k|)\log\left(\dfrac{|k|}{2\pi} - 1\right), & 2\pi < |k| < 4\pi \end{cases}$$

$$g_3(k) = \begin{cases} \dfrac{1}{2}\left(\dfrac{\pi^2}{6} - 2\operatorname{dilog}\left(\dfrac{2\pi}{|k|}\right)\right. \\ \qquad \left. - \log\left(\dfrac{2\pi}{|k|}\right)\log\left(\dfrac{(2\pi - |k|)^2}{2\pi\,|k|}\right)\right), & |k| < 2\pi \\ \dfrac{1}{2}\left(\operatorname{dilog}\left(\dfrac{|k| - 2\pi}{2\pi}\right) - \operatorname{dilog}\left(\dfrac{2\pi}{|k|}\right)\right. \\ \qquad \left. + \log\left(\dfrac{|k| - 2\pi}{2\pi}\right)\log\left(\dfrac{|k|}{2\pi}\right)\right), & 2\pi < |k| < 4\pi \end{cases}$$

$$g_4(k) = \begin{cases} -\dfrac{\pi^2}{6} - \text{dilog}\left(\dfrac{2\pi}{2\pi - |k|}\right) + 2\,\text{dilog}\left(\dfrac{4\pi - |k|}{2\pi - |k|}\right) \\ \quad + 2\log\left(\dfrac{2\pi}{2\pi - |k|}\right)\log\left(\dfrac{4\pi - |k|}{2\pi - |k|}\right), & |k| < 2\pi \\ \text{dilog}\left(\dfrac{|k|}{|k| - 2\pi}\right) - 2\,\text{dilog}\left(\dfrac{2\pi}{|k| - 2\pi}\right) \\ \quad + \log\left(\dfrac{2\pi}{|k| - 2\pi}\right)\log\left(\dfrac{|k|}{|k| - 2\pi}\right), & 2\pi < |k| < 4\pi \end{cases} \tag{7.21}$$

The above formula for $S(k; \beta)$ in the case $|k| < 2\pi$ (recall here $\rho = 1$) can be used to expand $\partial S(k; \beta)/\partial\beta$ about $k = 0$. For this task we use computer algebra, which gives the result

$$\left.\frac{\partial S(k; \beta)}{\partial\beta}\right|_{\beta=4} = -\frac{|k|}{16\pi} + \frac{|k|^3}{256\pi^3} + \frac{5k^4}{3072\pi^4} + \frac{3\,|k|^5}{4096\pi^5} + \frac{27k^6}{81920\pi^6} + \frac{37\,|k|^7}{245760\pi^7} + \frac{1273k^8}{18350080\pi^8} + \frac{887\,|k|^9}{27525120\pi^9} + \frac{4423k^{10}}{293601280\pi^{10}} + \frac{1949\,|k|^{11}}{275251200\pi^{11}} + \cdots \tag{7.22}$$

This allows us to deduce a further equation for $\{b_{8,\,j}\}_{j=0,\dots,4}$ and $\{\tilde{a}_{9,\,j}\}_{j=0,\dots,4}$, which in combination with (7.10), (6.10), (5.14), (5.9) and (5.10) implies

$$p_8(x) = (x-1)^2\,(1 - \tfrac{263}{84}x + \tfrac{1697}{315}x^2 - \tfrac{6337}{1008}x^3 + \tfrac{1697}{315}x^4 - \tfrac{263}{84}x^5 + x^6) \tag{7.23}$$

$$p_9(x) = (x-1)(1 - \tfrac{791}{180}x + \tfrac{73603}{7560}x^2 - \tfrac{7355}{504}x^3 + \tfrac{2231}{135}x^4 - \tfrac{7355}{504}x^5 + \tfrac{73603}{7560}x^6 - \tfrac{791}{180}x^7 + x^8) \tag{7.24}$$

8. CONCLUSION

Collecting together the evaluations (1.12), (3.15), (5.15), (4.19), (6.11), (7.11), (7.23) and (7.24), and substituting in (1.8) we have that for $|k| < \min(2\pi\rho, \pi\beta\rho)$

$$
\begin{aligned}
\frac{\pi\beta}{|k|}\, S(k;\beta) = 1 \\
&+ (x-1)\, y \\
&+ (x-1)^2\, y^2 \\
&+ (x-1)(x^2 - \tfrac{11}{6}x + 1)\, y^3 \\
&+ (x-1)^2\, (x^2 - \tfrac{3}{2}x + 1)\, y^4 \\
&+ (x-1)(x^4 - \tfrac{91}{30}x^3 + \tfrac{62}{15}x^2 - \tfrac{91}{30}x + 1)\, y^5 \\
&+ (x-1)^2\, (x^4 - \tfrac{37}{15}x^3 + \tfrac{13}{4}x^2 - \tfrac{37}{15}x + 1)\, y^6 \\
&+ (x-1)(x^6 - \tfrac{1607}{420}x^5 + \tfrac{2011}{280}x^4 - \tfrac{911}{105}x^3 + \tfrac{2011}{280}x^2 - \tfrac{1607}{420}x + 1)\, y^7 \\
&+ (x-1)^2\, (x^6 - \tfrac{263}{84}x^5 + \tfrac{1697}{315}x^4 - \tfrac{6337}{1008}x^3 + \tfrac{1697}{315}x^2 - \tfrac{263}{84}x + 1)\, y^8 \\
&+ (x-1)(x^8 - \tfrac{791}{180}x^7 + \tfrac{73603}{7560}x^6 - \tfrac{7355}{504}x^5 + \tfrac{2231}{135}x^4 \\
&- \tfrac{7355}{504}x^3 + \tfrac{73603}{7560}x^2 - \tfrac{791}{180}x + 1)\, y^9 \\
&+ O(y^{10}) \qquad (8.1)
\end{aligned}
$$

where $x = \beta/2$ and $y = |k|/\pi\beta\rho$. With the coefficient of y^j denoted $p_j(x)$ as has been throughout, we recall from our workings above that $p_0(x)$, $p_1(x)$, $p_2(x)$ and $p_4(x)$ have been calculated for general values of β. In all other cases the calculation has relied on the assumption that the $p_j(x)$ are indeed polynomials. On this point we remark that in such cases, excluding $j=8$ and 9, we have more data points than is necessary to uniquely specify $p_j(x)$, assuming it is a polynomial, and our extra data points are consistent with the explicit forms presented in (8.1).

We remark that the structure exhibited by (8.1) is familiar from the study of exactly solvable two-dimensional lattice models.[10] In this field one encounters two-variable generating functions $G(x, y)$ say with series expansions of the form

$$G(x, y) = \sum_{n=0}^{\infty} H_n(x)\, y^n \qquad (8.2)$$

in which $H_n(x)$ is a rational function, and furthermore the denominator polynomial in $H_n(x)$ only has a small number of (typically no more than two) distinct zeros. For example, the two-dimensional Ising model with couplings J_1 (J_2) between bonds in the horizontal (vertical) direction and

$x := \exp(-4J_1/k_B T)$, $y := \exp(-4J_2/k_B T)$ has for its spontaneous magnetization the celebrated exact expression (see e.g., ref. 2)

$$M(x, y) = \left(1 - \frac{16xy}{(1-x)^2\,(1-y)^2}\right)^{1/8} \tag{8.3}$$

When written in the form (8.2) one finds

$$H_n(x) = \frac{2xP_n(x)}{(1-x)^n} \tag{8.4}$$

where $P_n(x)$ is a polynomial of degree $2n-2$ which satisfies the functional relation

$$P_n(x) = x^{2n-2} P_n(1/x) \tag{8.5}$$

As emphasized in ref. 10, the exact solution (8.3) can be uniquely determined by the functional form (8.4), together with the functional (inversion) relation (8.5) and the symmetry relation $M(x, y) = M(y, x)$. For the structure function of the log–gas we have no analogue of the symmetry relation and so cannot characterize (2.6) this way.

One immediate feature of the polynomials $p_j(x)$ in (8.1) is that for j even the polynomial $p_j(-x)$ has all coefficients positive, while for j odd the polynomial $p_j(-x)$ has all coefficients negative. Another general feature of the $p_j(x)$ in (8.1), obtained from numerical computation, is that all the zeros lie on the unit circle in the complex x-plane. This can be rigorously determined numerically because the symmetry (1.9) implies that if x_0 is a zero of $p_j(x)$, then so is $1/x_0$, which will be the complex conjugate of x_0 if and only if $|x_0| = 1$.

The quantum many body interpretation of (1.1) allows us to give a physical interpretation to the functional relation (1.7). As the functional relation is derived from the integral representation (2.9), it is appropriate to recall[11] the physical interpretation of that formula. In (2.9), with $\beta/2 = p/q$, there are q integrals over $x_i \in (0, \infty)$ and p integrals over $y_j \in (0, 1)$. The variables x_i can be interpreted as being rapidities of quasi-particle excitations, while the y_j are rapidities of quasi-hole excitations. Thus the transformation $\beta \mapsto 4/\beta$ is equivalent to interchanging p and q and thus the quasi-holes and quasi-particles. In (2.9) this does not lead to an integral of the same functional form as before; although the functional form of the integrand is conserved, apart from a renormalization of k, the domain of integration is different for $\{x_i\}$ and $\{y_j\}$. But with k restricted as in (2.8) both sets of variables can take any value in $(0, \infty)$. The quasi-particles and quasi-holes play an identical role and the functional equation results.

It is of interest to consider the small k expansion of $S(k; \Gamma)$, $\Gamma := q^2/k_B T$ ($q =$ charge), for the two-dimensional one-component plasma. As mentioned earlier, this has recently been the object of study of Kalinay *et al.*[16] They obtain results which imply

$$\frac{2\pi\Gamma}{k^2} S(k; \Gamma) = 1 + \left(\frac{\Gamma}{4} - 1\right) \frac{k^2}{2\pi\Gamma\rho} + \left(\frac{\Gamma}{4} - \frac{3}{2}\right)\left(\frac{\Gamma}{4} - \frac{2}{3}\right)\left(\frac{k^2}{2\pi\Gamma\rho}\right)^2 + O(k^6) \tag{8.6}$$

where $k := |\vec{k}|$. The structure of (8.6) bears a striking resemblence to (8.1) with $\Gamma/4$ corresponding to x and $k^2/2\pi\Gamma\rho$ to y. In particular with $g(x, y) := (2\pi\Gamma/k^2)\, S(k; \Gamma)$, the expansion (8.6) to the given order is such that

$$g(x, y) = g\left(\frac{1}{x}; -yx\right) \tag{8.7}$$

Furthermore, writing

$$g(x, y) = 1 + \sum_{l=1}^{\infty} u_l(x)\, y^l \tag{8.8}$$

we have $u_1(x) = (x-1)$, $u_2(x) = (x - 3/2)(x - 2/3)$ so $u_l(x)$ is a monic lth degree polynomial for $l \leqslant 2$. However we can demonstrate that this analogy breaks down for the $l = 3$ term in (8.8).

To demonstrate this fact, suppose instead that the functional equation (8.8) was valid at order $l = 3$ in (8.8) and $u_3(x)$ is a monic polynomial. Then u_3 must be of the form

$$u_3(x) = (x-1)(x^2 + ax + 1) \tag{8.9}$$

From the definition of $g(x, y)$ we can check that this is equivalent to the statement that

$$\frac{1}{\rho}\left(\frac{\pi\Gamma\rho}{2}\right)^4 \int_{\mathbf{R}^2} r^8 S(r; \Gamma)\, d\vec{r} = (4!)^2\, (x-1)(x^2 + ax + 1) \tag{8.10}$$

But as noted in ref. 16, it follows from the perturbation expansion of ref. 13 that

$$\frac{1}{\rho}\left(\frac{\pi\Gamma\rho}{2}\right)^4 \int_{\mathbf{R}^2} r^8 S(r;\Gamma)\, d\vec{r} = -4! + (\Gamma-2)\, 4! \left(\sum_{k=0}^{4} \frac{2^k-1}{k+1} - 2\right) + O((\Gamma-2)^2)$$
$$= -4! + (\Gamma-2)\, 4!\, \frac{17}{4} + O((\Gamma-2)^2) \qquad (8.11)$$

The term in (8.11) proportional to $\Gamma-2$ is incompatible with (8.10) which gives instead

$$(\Gamma-2)\, 4!\tfrac{18}{4}$$

independent of the value of a. Indeed in ref. 16 evidence is presented which indicates $u_3(x)$ is an infinite series in x, although we have no way of determining if the functional equation (8.7) also breaks down at this order.

APPENDIX A

In this appendix some details of the derivation of (7.7) and (7.9) will be given. To simplify notation we take $\rho = 1$ throughout. The first step is to substitute (7.2) and (7.1) and simplify by expanding out the determinant and cancelling terms where possible. This shows that up to terms $O((\beta-2)^2)$

$$\begin{aligned}
&g_2(x_1, x_2; \beta)\\
&\quad = 1 - (P_2(x_1, x_2))^2 + (\beta-2)\left\{ -(1-(P_2(x_1,x_2))^2)\, \Phi(x_1, x_2) \right.\\
&\qquad -2\int_{-\infty}^{\infty} (-(P_2(x_2,x_3))^2 - (P_2(x_1,x_3))^2\\
&\qquad + 2P_2(x_1,x_2)\, P_2(x_2,x_3)\, P_2(x_3,x_1))\, \Phi(x_1,x_3)\, dx_3\\
&\qquad -\tfrac{1}{2}\int_{-\infty}^{\infty} (4P_2(x_1,x_3)\, P_2(x_3,x_4)\, P_2(x_4,x_1)\\
&\qquad -4P_2(x_1,x_2)\, P_2(x_2,x_3)\, P_2(x_3,x_4)\, P_2(x_4,x_1)\\
&\qquad -2P_2(x_1,x_3)\, P_2(x_3,x_2)\, P_2(x_2,x_4)\, P_2(x_4,x_1)\\
&\qquad \left. + 2(P_2(x_1,x_3))^2\, (P_2(x_2,x_4))^2))\, \Phi(x_3,x_4)\, dx_3\, dx_4 \right\}
\end{aligned} \qquad (A.1)$$

The convolution structure

$$\int_{-\infty}^{\infty} f(y_1-x)\, g(x-y_2)\, dx$$

often occurs in the above integrals. Such an integral can be transformed by introducing the Fourier transforms $\hat{f}$ $(\hat{g})$ according to the formula

$$\int_{-\infty}^{\infty} f(y_1-x)\, g(x-y_2)\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(l)\, \hat{g}(l)\, e^{-il(y_1-y_2)}\, dl \tag{A.2}$$

Making use of this formula typically leads to simplifications.

For example, consider the first integral in (A.1). Starting with the Fourier transform

$$\int_{-\infty}^{\infty} \frac{\sin^2 \pi x}{(\pi x)^2} e^{ikx}\, dx = \begin{cases} 1-\dfrac{|k|}{2\pi}, & |k|<2\pi \\ 0, & |k|>2\pi \end{cases} \tag{A.3}$$

and (5.12), application of (A.2) gives

$$A_1(x_{12}) := \int_{-\infty}^{\infty} (P_2(x_2, x_3))^2\, \Phi(x_3, x_1)\, dx_3 = \int_{-2\pi}^{2\pi} \left(1-\frac{|k|}{2\pi}\right) \frac{\pi}{|k|} \cos kx_{12} \frac{dk}{2\pi} \tag{A.4}$$

This expression is indeed simpler than the original, but it suffers from being ill-defined, due to the singularity at the origin. However its derivative is well-defined, and can furthermore be evaluated in terms of elementary functions giving

$$\frac{d}{dx} A_1(x) = \frac{\sin 2\pi x}{2\pi x^2} - \frac{1}{x} \tag{A.5}$$

Also, we have[9]

$$A_1(0) = -\int_{-\infty}^{\infty} dx\, \frac{\sin^2 \pi x}{(\pi x)^2} \log|x| = C + \log 2\pi - 1 \tag{A.6}$$

where C denotes Euler's constant. Together (A.5) and (A.6) imply

$$A_1(x) = -\frac{\sin 2\pi x}{2\pi x} + \mathrm{ci}(2\pi x) - \log|x| \tag{A.7}$$

where $\mathrm{ci}(x)$ denotes the cosine integral (7.8).

The other six integrals in (A.1) yield to similar techniques. We find

$$A_2 := \int_{-\infty}^{\infty} (P_2(x_1, x_3))^2\, \Phi(x_1, x_3)\, dx_3 = A_1(0)$$

$$\begin{aligned}
A_3(x_{12}) &:= \int_{-\infty}^{\infty} P_2(x_2, x_3)\, P_2(x_3, x_1)\, \Phi(x_3, x_1)\, dx_3 \\
&= \int_{-\pi}^{\pi} (C + \log(\pi + k)) \cos kx_{12} \frac{dk}{2\pi} \\
&= \frac{1}{2}(C + \log 2\pi - \log|x_{12}| + \mathrm{ci}(2\pi x_{12})) \frac{\sin \pi x_{12}}{\pi x_{12}} \\
&\quad - \frac{1}{2}\left(\mathrm{si}(2\pi x_{12}) + \frac{\pi}{2}\right) \frac{\cos \pi x_{12}}{\pi x_{12}}
\end{aligned}$$

$$\begin{aligned}
A_4 &:= \int_{-\infty}^{\infty} P_2(x_1, x_3)\, P_2(x_3, x_4)\, P_2(x_4, x_1)\, \Phi(x_3, x_4)\, dx_3\, dx_4 \\
&= A_3(0) = A_1(0)
\end{aligned}$$

$$A_5(x_{12}) := \int_{-\infty}^{\infty} P_2(x_2, x_3)\, P_2(x_3, x_4)\, P_2(x_4, x_1)\, \Phi(x_3, x_4)\, dx_3\, dx_4 = A_3(x_{12})$$

$$\begin{aligned}
A_6(x_{12}) &:= \int_{-\infty}^{\infty} P_2(x_1, x_3)\, P_2(x_3, x_2)\, P_2(x_2, x_4)\, P_2(x_4, x_1)\, \Phi(x_3, x_4)\, dx_3\, dx_4 \\
&= \frac{1}{2(\pi x_{12})^2}\Bigg(C + \log 2\pi + \cos 2\pi x_{12}(\log|x_{12}| - \mathrm{ci}(2\pi x_{12})) \\
&\quad - \sin 2\pi x_{12}\left(\mathrm{si}(2\pi x_{12}) + \frac{\pi}{2}\right)\Bigg)
\end{aligned}$$

$$\begin{aligned}
A_7(x_{12}) &:= \int_{-\infty}^{\infty} (P_2(x_1, x_3))^2\, (P_2(x_2, x_4))^4\, \Phi(x_3, x_4)\, dx_3\, dx_4 \\
&= \int_{-2\pi}^{2\pi} \frac{dk}{2\pi}\left(1 - \frac{|k|}{2\pi}\right)^2 \frac{\pi}{|k|} \cos kx_{12} \\
&= -\log|x_{12}| - \frac{1 - \cos 2\pi x_{12}}{(2\pi x_{12})^2} - \frac{\sin 2\pi x_{12}}{2\pi x_{12}} + \mathrm{ci}(2\pi x_{12})
\end{aligned} \tag{A.8}$$

where $\mathrm{si}(x)$ denotes the sine integral defined in (7.5).

Of the results (A.8), the evaluation of A_6 is the most difficult, so it is appropriate to give details in that case also. We observe that A_6 consists of the convolution of $P_2(x_1, x_3)\, P_2(x_3, x_2)$ regarded as a function of x_3, and $\Phi(x_3, x_4)$, and $P_2(x_4, x_1)\, P_2(x_2, x_4)$ regarded as a function of x_4. It simplifies the calculation to take as the origin in both integrations the centre of the interval between particle 1 and particle 2, which is achieved by the change of variables $x_3 \mapsto x_3 + (x_1 + x_2)/2$, $x_4 \mapsto x_4 + (x_1 + x_2)/2$. Use of (A.2) then shows

$$A_6(x_{12}) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} (\hat{V}(k, x_{12}))^2 \frac{\pi}{|k|} \tag{A.9}$$

$$\begin{aligned} \hat{V}(k, x_{12}) &:= \int_{-\infty}^{\infty} \frac{\sin \pi x_{13}}{\pi x_{13}} \frac{\sin \pi x_{32}}{\pi x_{32}} \cos k x_3 \, dx_3 \\ &= \begin{cases} \dfrac{1}{\pi x_{12}} \sin\left(\pi - \dfrac{|k|}{2}\right) x_{12}, & |k| < 2\pi \\ 0, & |k| > 2\pi \end{cases} \end{aligned} \tag{A.10}$$

where the second equality in (A.10) follows after further use of (A.2). Thus

$$A_6(x) = \int_{-2\pi}^{2\pi} \frac{dk}{2\pi} \left(\frac{\sin(\pi - |k|/2)\, x}{\pi x}\right)^2 \frac{\pi}{|k|} \tag{A.11}$$

As in (A.11), this integrand is ill-defined. To proceed further, we write

$$A_6(x) = A_6^{(1)}(x) + A_6^{(2)}(x)$$

where

$$A_6^{(1)}(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left\{ \left(\frac{\sin(\pi - |k|/2)\, x}{\pi x}\right)^2 - \left(\frac{\sin \pi x}{\pi x}\right)^2 \right\} \frac{\pi}{|k|}$$

$$A_6^{(2)}(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2 \int_{-2\pi}^{2\pi} \frac{dk}{2\pi} \frac{\pi}{|k|}$$

The integral defining $A_6^{(1)}$ is well defined and can be computed by elementary means. The integral defining $A_6^{(2)}$ is singular. It coincides with the singular part of $A_1(0)$ (recall (A.4)), and so from (A.6) we have

$$A_6^{(1)}(0) = \left(\frac{\sin \pi x}{\pi x}\right)^2 (C + \log 2\pi)$$

Collecting together the above evaluations of A_1–A_7 and substituting as appropriate in (A.1) gives (7.7).

The next task is to evaluate the Fourier transform. Now the evaluations of A_1 and A_7 are given as Fourier integrals, so their Fourier transform is immediate:

$$\begin{aligned} \mathrm{FT}\ A_1(x) &= \frac{\pi}{|k|} - \frac{1}{2}, \qquad && |k| < 2\pi \\ \mathrm{FT}\ A_7(x) &= \frac{\pi}{|k|} - 1 + \frac{|k|}{4\pi}, \qquad && |k| < 2\pi \end{aligned} \tag{A.12}$$

while for $|k| > 2\pi$

$$\mathrm{FT}\ A_1(x) = \mathrm{FT}\ A_7(x) = 0 \tag{A.13}$$

We can check that the constants A_2 and A_4 cancel when substituted in (A.1), and so play no further part in the calculation.

Of the remaining terms, consider first the first term proportional to $\beta - 2$ in (A.1), $A_0(x)$ say. Making use of (A.2) we see that

$$\mathrm{FT}\ A_0(x) = -\frac{\pi}{|k|} + \int_{-2\pi}^{2\pi} \frac{dl}{|l-k|}\left(1 - \frac{|l|}{2\pi}\right) \tag{A.14}$$

For $|k| < 2\pi$ minor manipulation allows the singular part

$$\int_{-2\pi}^{2\pi} \frac{dl}{2\pi}\frac{\pi}{|l|} = C + \log 2\pi \tag{A.15}$$

to be separated, while the remaining convergent integrals are elementary. We thus find that for $|k| < 2\pi$

$$\begin{aligned} \mathrm{FT}\ A_0(x) = &-\frac{\pi}{|k|} + \left\{C + \log 2\pi + \frac{1}{2}\log\left(1 - \left(\frac{k}{2\pi}\right)^2\right)\right\}\left(1 - \frac{|k|}{2\pi}\right) - 1 \\ &+ \frac{|k|}{2\pi} + \frac{|k|}{2\pi}\log\left(\frac{2\pi + |k|}{|k|}\right) \end{aligned} \tag{A.16}$$

For $|k| > 2\pi$ the integrals in (A.14) are convergent and also elementary. In this case we find

$$\mathrm{FT}\ A_0(x) = -\frac{\pi}{|k|} + \frac{1}{2}\log\frac{|k| + 2\pi}{|k| - 2\pi} + \frac{|k|}{4\pi}\log\left(1 - \frac{4\pi^2}{k^2}\right) \tag{A.17}$$

To compute the Fourier transform of A_6, we begin by making use of (A.2) in (A.9) thereby obtaining

$$\mathrm{FT}\, A_6 = \int_{-\infty}^{\infty} \frac{dl}{2\pi} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\, \hat{V}(l, k_1)\, \hat{V}(l, -(k_1 - k)) \frac{\pi}{|l|}$$

where

$$\hat{V}(l, k) := \int_{-\infty}^{\infty} dx\, \hat{V}(l, x)\, e^{\pi i k x} = \chi_{|k| < \pi - |l|/2}$$

with the equality in the latter formula following from the explicit form (A.10) of $\hat{V}(l, x)$ and then computation of the resulting integral, and where $\chi_T = 1$ for T true and $\chi_T = 0$ otherwise. Thus

$$\begin{aligned} \mathrm{FT}\, A_6(x) &= \int_{-\infty}^{\infty} \frac{dl}{2\pi} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\, \chi_{|k_1| < \pi - |l|/2}\, \chi_{|k_1 - k| < \pi - |l|/2} \frac{\pi}{|l|} \\ &= \begin{cases} \left(1 - \dfrac{|k|}{2\pi}\right)(C + \log 2\pi) - \left(1 - \dfrac{|k|}{2\pi}\right) \log \dfrac{2\pi}{2\pi - |k|} \\ \qquad - \dfrac{1}{2\pi}(2\pi - |k|), \qquad |k| < 2\pi \\ 0, \qquad |k| > 2\pi \end{cases} \end{aligned} \tag{A.18}$$

where use has been made of the generalized integral evaluation (A.15).

The final Fourier transform to consider is

$$\begin{aligned} &\mathrm{FT}\, \frac{\sin \pi x}{\pi x} A_3(x) \\ &\quad = \mathrm{FT}\, \frac{\sin \pi x}{\pi x} \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \left(C + \frac{1}{2}\log(\pi + k_1) + \frac{1}{2}\log(\pi - k_1)\right) e^{ik_1 x} \\ &\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} dl\, \chi_{l \in [-\pi, \pi]}\, \chi_{l \in [-\pi + k, \pi + k]} \\ &\qquad \times \left(C + \frac{1}{2}\log(\pi + l - k) + \frac{1}{2}\log(\pi - (l - k))\right) \end{aligned}$$

where to obtain the equality use has been made of (A.2). Evaluating the integral gives

$$\mathrm{FT}\,\frac{\sin \pi x}{\pi x}\,A_3(x) = (C + \log 2\pi)\left(1 - \frac{|k|}{2\pi}\right) - \frac{|k|}{4\pi}\log\frac{|k|}{2\pi} + \frac{1}{2}\left(1 - \frac{|k|}{2\pi}\right)\log\left(1 - \frac{|k|}{2\pi}\right) \tag{A.19}$$

for $|k| < 2\pi$, while for $|k| > 2\pi$

$$\mathrm{FT}\,\frac{\sin \pi x}{\pi x}\,A_3(x) = 0 \tag{A.20}$$

Substituting the above results as appropriate in the Fourier transform of (A.1) gives the result (7.9).

APPENDIX B

In this appendix we outline some details of the calculation of (7.1) in the case $\beta_0 = 4$ and show how this leads to (7.13). Because (7.2) and (7.3) formally have the same structure upon expansion (recall the definition of qdet (7.6)), the formula (A.1) formally maintains its structure when generalized to the case $\beta_0 = 4$. Thus we have

$$\begin{aligned}
g_2(x_1, x_2; \beta) &= 1 - (P_4(x_1, x_2)\, P_4(x_2, x_1))^{(0)} \\
&\quad + (\beta - 4)\Big\{ -(1 - (P_4(x_1, x_2)\, P_4(x_2, x_1))^{(0)})\, \Phi(x_1, x_2) \\
&\quad - 2\int_{-\infty}^{\infty} (-(P_4(x_2, x_3)\, P_4(x_3, x_2))^{(0)} - (P_4(x_1, x_3)\, P_4(x_3, x_1))^{(0)} \\
&\quad + 2(P_4(x_1, x_2)\, P_4(x_2, x_3)\, P_4(x_3, x_1))^{(0)})\, \Phi(x_1, x_3)\, dx_3 \\
&\quad - \tfrac{1}{2}\int_{-\infty}^{\infty} (4(P_4(x_1, x_3)\, P_4(x_3, x_4)\, P_4(x_4, x_1))^{(0)} \\
&\quad - 4(P_4(x_1, x_2)\, P_4(x_2, x_3)\, P_4(x_4, x_1))^{(0)} \\
&\quad - 2(P_4(x_1, x_3)\, P_4(x_3, x_2)\, P_4(x_2, x_4)\, P_4(x_4, x_1))^{(0)} \\
&\quad + 2(P_4(x_1, x_3)\, P_4(x_3, x_1))^{(0)}\, (P_4(x_2, x_4)\, P_4(x_4, x_2))^{(0)}) \\
&\quad \times \Phi(x_3, x_4)\, dx_3\, dx_4 \Big\} + O((\beta - 4)^2)
\end{aligned} \tag{B.1}$$

We treat each of the seven distinct integrals in (B.1) in an analogous way to their counterparts in (A.1), although extra working is involved due to P_4 being a matrix rather than a scalar.

The final results are

$$B_1(x_{12}) := \int_{-\infty}^{\infty} (P_4(x_2, x_3)\, P_4(x_3, x_2))^{(0)}\, \Phi(x_1, x_3)\, dx_3$$

$$= \int_{-4\pi}^{4\pi} \frac{dk}{2\pi} \left(1 - \frac{|k|}{4\pi} + \frac{|k|}{8\pi} \log \left|1 - \frac{|k|}{2\pi}\right|\right) \frac{\pi}{|k|} \cos kx_{12}$$

$$= -\log |x_{12}| - \frac{\sin 4\pi x_{12}}{4\pi x_{12}} + \mathrm{ci}(4\pi x_{12}) - \frac{\cos 2\pi x_{12}}{4\pi x_{12}} \mathrm{Si}(2\pi x_{12})$$

$$B_2 := \int_{-\infty}^{\infty} (P_4(x_1, x_3)\, P_4(x_3, x_1))^{(0)}\, \Phi(x_1, x_3)\, dx_3 = B_1(0)$$

$$= C + \log 4\pi - \frac{3}{2}$$

$$B_3(x_{12}) := \int_{-\infty}^{\infty} P_4(x_2, x_3)\, P_4(x_3, x_1)\, \Phi(x_1, x_3)\, dx_3$$

$$= \begin{bmatrix} \frac{1}{4} f_1(x_{12}) - \frac{1}{4} f_3(x_{12}) & -\frac{1}{4} \int_0^{x_{12}} (f_1(t) + f_2(t))\, dt \\ -\frac{1}{4} f'_1(x_{12}) + \frac{1}{4} f'_3(x_{12}) & \frac{1}{4} f_1(x_{12}) + \frac{1}{4} f_2(x_{12}) \end{bmatrix}$$

$$B_4 := \int_{-\infty}^{\infty} (P_4(x_1, x_3)\, P_4(x_3, x_4)\, P_4(x_4, x_1))^{(0)}\, \Phi(x_1, x_3)\, dx_3\, dx_4$$

$$= B_1(0)$$

$$B_5(x_{12}) := \int_{-\infty}^{\infty} P_4(x_2, x_3)\, P_4(x_3, x_4)\, P_4(x_4, x_1)\, \Phi(x_3, x_4)\, dx_3\, dx_4$$

$$= \begin{bmatrix} \frac{1}{4} f_1(x_{12}) + \frac{1}{8} f_2(x_{12}) - \frac{1}{8} f_3(x_{12}) & -\int_0^{x_{12}} (\frac{1}{4} f_1(t) + \frac{1}{8} f_2(t) - \frac{1}{8} f_3(t))\, dt \\ -(\frac{1}{4} f'_1(x_{12}) + \frac{1}{8} f'_2(x_{12}) - \frac{1}{8} f'_3(x_{12})) & \frac{1}{4} f_1(x_{12}) + \frac{1}{8} f_2(x_{12}) - \frac{1}{8} f_3(x_{12}) \end{bmatrix}$$

$$B_6(x_{12}) := \int_{-\infty}^{\infty} (P_4(x_1, x_3)\, P_4(x_3, x_2)\, P_4(x_2, x_4)\, P_4(x_4, x_3))^{(0)}$$

$$\times \Phi(x_3, x_4)\, dx_3\, dx_4$$

$$
\begin{aligned}
&= \int_{-4\pi}^{4\pi} \frac{1}{2\,|k|} \Big\{ (g_1(k, x_{12}))^2 + \cos(kx_{12}/2)\; g_1(k, x_{12}) \\
&\quad \times \left(\frac{|k|}{4\pi}\, g_2(k, x_{12}) - \frac{ik}{4\pi}\, g_3(k, x_{12}) \right) \\
&\quad + \cos(kx_{12}) \left(\frac{|k|}{8\pi}\, g_2(k, x_{12}) - \frac{ik}{8\pi}\, g_3(k, x_{12}) \right)^2 \\
&\quad - \left(\frac{\sin(|k|\, x_{12}/2)}{4\pi}\, g_2(k, x_{12}) + \frac{\cos(kx_{12}/2)}{4\pi}\, g_3(k, x_{12}) \right) \\
&\quad \times \left(\frac{(4\pi - |k|)\cos((2\pi - |k|/2))\, x_{12})}{4\pi x_{12}} \right. \\
&\quad \left. - \frac{\sin(2\pi - |k|/2)\, x_{12})}{2\pi x_{12}^2} \right) \Big\}\, dk
\end{aligned}
$$

$$
\begin{aligned}
B_7(x_{12}) &:= \int_{-\infty}^{\infty} (P_4(x_1, x_3)\, P_4(x_3, x_1))^{(0)} (P_4(x_2, x_4)\, P_4(x_4, x_2))^{(0)} \\
&\quad \times \Phi(x_3, x_4)\, dx_3\, dx_4 \\
&= \int_{-4\pi}^{4\pi} \left(1 - \frac{|k|}{4\pi} + \frac{|k|}{8\pi} \ln \left| 1 - \frac{|k|}{2\pi} \right| \right)^2 \frac{\pi}{|k|} \cos kx_{12}\, \frac{dk}{2\pi} \\
&= -\log |x| - \frac{\sin 4\pi x}{4\pi x} + \mathrm{ci}(4\pi x) - \frac{\mathrm{Si}(2\pi x)\sin 2\pi x}{8\pi^2 x^2} \\
&\quad - \frac{\mathrm{Si}(2\pi x)\cos 2\pi x}{4\pi x} \\
&\quad + \int_{-4\pi}^{4\pi} |k| \log \left| 1 - \frac{|k|}{2\pi} \right|^2 \cos kx\, \frac{dk}{128\pi^2}
\end{aligned} \tag{B.2}
$$

where

$$
\begin{aligned}
f_1(x) &:= \int_{-2\pi}^{2\pi} (2C + \ln(4\pi^2 - k^2)) \cos kx\, \frac{dk}{2\pi} \\
&= \frac{\sin 2\pi x}{\pi x} (C + \ln 4\pi - \ln |x| + \mathrm{ci}(4\pi x)) - \frac{\cos 2\pi x}{\pi x}\, \mathrm{Si}(4\pi x) \\
f_2(x) &:= -\frac{1}{\pi x}\, \mathrm{Si}(2\pi x) \\
f_3(x) &:= \frac{\sin 2\pi x}{\pi x}
\end{aligned}
$$

$$g_1(k,x)=\frac{\sin((2\pi-|k|/2)\,x)}{2\pi x}$$

$$g_2(k,x)=\mathrm{ci}((2\pi-|k|)\,x)-\mathrm{ci}(2\pi x)$$

$$g_3(k,x)=\mathrm{Si}((2\pi-|k|)\,x)+\mathrm{Si}(2\pi x) \tag{B.3}$$

When substituted in (B.1), the constant terms B_2 and B_4 cancel, and we obtain the formula

$$\begin{aligned}g(x_1,x_2;\beta)=1&-(P_4(x_1,x_2)\,P_4(x_2,x_1))^{(0)}\\&+(\beta-4)\{(-1+(P_4(x_1,x_2)\,P_4(x_2,x_1))^{(0)}\,\Phi(x_1,x_2)\\&+2B_1(x_{12})-4(P_4(x_{12})\,B_3(x_{12}))^{(0)}+2(P_4(x_{12})\,B_5(x_{12}))^{(0)}\\&+B_6(x_{12})-B_7(x_{12})\}+O((\beta-4)^2)\end{aligned} \tag{B.4}$$

We will demonstrate the close analogy with the $\beta=2$ calculation of Appendix A by giving the derivation of the integral formula in (B.2) for $B_1(x_{12})$. As in the derivation of the integral formula (A.4) for $A_1(x_{12})$, our strategy is to use the convolution formula (A.2). However here the Fourier transform of $P_4(x_1,x_2)\,P_4(x_2,x_1)$ is not immediate. What is immediate is the Fourier transform of $P_4(x_1,x_2)$. Thus from the definition (7.4) we see that

$$\begin{aligned}\mathrm{FT}\,P_4(x_1,x_2)\;&:=\int_{-\infty}^{\infty}P_4(x_1,x_2)\,e^{ikx_{12}}\,dx_{12}\\&=\begin{cases}\begin{bmatrix}1/2 & i/2k\\ -ik/2 & 1/2\end{bmatrix}, & |k|<2\pi\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, & |k|>2\pi\end{cases}\end{aligned}$$

Use of (A.2) then shows that for $|k|<4\pi$

$$\begin{aligned}&\mathrm{FT}\,P(x_1,x_2)\,P_4(x_2,x_1)\\&\quad=\int_{-2\pi}^{2\pi}\begin{bmatrix}1/2 & i/2l\\ -il/2 & 1/2\end{bmatrix}\begin{bmatrix}1/2 & -i/2(k-l)\\ i(k-l)/2 & 1/2\end{bmatrix}\chi_{|k-l|<2\pi}\,\frac{dl}{2\pi}\\&\quad=\begin{bmatrix}1-\dfrac{|k|}{4\pi}+\dfrac{|k|}{8\pi}\log\left|1-\dfrac{|k|}{2\pi}\right| & 0\\ 0 & 1-\dfrac{|k|}{4\pi}+\dfrac{|k|}{8\pi}\log\left|1-\dfrac{|k|}{2\pi}\right|\end{bmatrix}\end{aligned} \tag{B.5}$$

while for $|k| > 4\pi$

$$\mathrm{FT}\ P_4(x_1, x_2)\ P_4(x_2, x_1) = 0 \tag{B.6}$$

The results (B.5) and (B.6) are the analogue of (A.3) in the working leading to the evaluation of $A_1(x_{12})$. The integral formula for $B_1(x_{12})$ in (B.2) now follows from (B.5), (B.6) and (5.12) upon a further application of (A.2).

The Fourier transform of (B.4) can be computed explicitly. The final result has already been stated in (7.13). This is obtained through the intermediate results

$$\mathrm{FT}\ B_j(x) = \hat{B}_j(k) \qquad \text{for} \quad j = 0, 1, 3, 5, 6, 7$$

with

$$B_0(x_{12}) := (P_4(x_1, x_2)\ P_4(x_2, x_1))^{(0)}\ \Phi(x_1, x_2)$$

and the $\hat{B}_j$ specified by (7.14)–(7.20). We will illustrate the working by giving some details of the computation of $\hat{B}_0(k)$ for $|k| < 4\pi$.

Using (B.5) and (5.12) we see from (A.2) that

$$\begin{aligned} \mathrm{FT}\ B_0(x_{12}) &= \int_{-4\pi}^{4\pi} \left(1 - \frac{|l|}{4\pi} + \frac{|l|}{8\pi} \log\left|1 - \frac{|l|}{2\pi}\right|\right) \frac{\pi}{|k-l|} \frac{dl}{2\pi} \\ &= \left(1 - \frac{|k|}{4\pi} + \frac{|k|}{8\pi} \log\left|1 - \frac{|k|}{2\pi}\right|\right) \int_{-4\pi}^{4\pi} \frac{\pi}{|k-l|} \frac{dl}{2\pi} \\ &\quad + \int_{-4\pi}^{4\pi} \left(\frac{|k| - |l|}{4\pi}\right) \frac{\pi}{|k-l|} \frac{dl}{2\pi} \\ &\quad + \int_{-4\pi}^{4\pi} \left(\frac{|l| - |k|}{8\pi}\right) \log\left|1 - \frac{|l|}{2\pi}\right| \frac{\pi}{|k-l|} \frac{dl}{2\pi} \\ &\quad + \frac{|k|}{8\pi} \int_{-4\pi}^{4\pi} \log\left|\frac{2\pi - |l|}{2\pi - |k|}\right| \frac{\pi}{|k-l|} \frac{dl}{2\pi} \end{aligned} \tag{B.7}$$

where the second equality, which follows from minor manipulation of the first integral, is motivated by the desire to separate the singular integral. Thus in the second equality of (B.7) only the first integral is singular. It is essentially the same as the first singular integral in (A.14), and is evaluated as

$$\int_{-4\pi}^{4\pi} \frac{\pi}{|k-l|} \frac{dl}{2\pi} = C + \frac{1}{2} \log(16\pi^2 - k^2), \qquad |k| \leqslant 4\pi \tag{B.8}$$

The second integral in the second equality of (B.7) also appears in the evaluation of (A.14). An elementary calculation shows

$$\int_{-4\pi}^{4\pi} \frac{|k|-|l|}{4\pi} \frac{\pi}{|k-l|} \frac{dl}{2\pi} = -1 + \frac{|k|}{4\pi} + \frac{|k|}{4\pi} \log\left(\frac{4\pi+|k|}{|k|}\right) \tag{B.9}$$

To evaluate the third integral in (B.7) we suppose without loss of generality that $k>0$ and write

$$\begin{aligned}\int_{-4\pi}^{4\pi} &\frac{|l|-|k|}{8\pi} \log\left|1-\frac{|l|}{2\pi}\right| \frac{\pi}{|k-l|} \frac{dl}{2\pi} \\ &= \int_0^{2\pi} \left(\frac{1}{16\pi} + \frac{k}{8\pi(l-k)}\right) \log\left|1+\frac{l}{2\pi}\right| dl \\ &\quad - \int_0^k \log\left|1-\frac{|l|}{2\pi}\right| \frac{dl}{16\pi} + \int_k^{4\pi} \log\left|1-\frac{|l|}{2\pi}\right| \frac{dl}{16\pi}\end{aligned} \tag{B.10}$$

The only non-elementary integral is the second term of the first integral. This can be computed by checking from the definition (7.12) that for $-4\pi<l<0$

$$\frac{d}{dl}\left(\operatorname{dilog}\left(\frac{k-l}{k+2\pi}\right) + \log\left|1+\frac{l}{2\pi}\right| \log\left(\frac{k-l}{k+2\pi}\right)\right) = \frac{1}{l-k} \log\left|1+\frac{l}{2\pi}\right| \tag{B.11}$$

In total we therefore have

$$\begin{aligned}\int_{-4\pi}^{4\pi} &\frac{|l|-|k|}{8\pi} \log\left|1-\frac{|l|}{2\pi}\right| \frac{\pi}{|k-l|} \frac{dl}{2\pi} \\ &= -\frac{1}{2} + \frac{|k|}{8\pi} + \frac{|k|}{8\pi}\left(\operatorname{dilog}\left(\frac{|k|}{2\pi+|k|}\right) - \operatorname{dilog}\left(\frac{4\pi+|k|}{2\pi+|k|}\right)\right) \\ &\quad + \frac{2\pi-|k|}{8\pi} \log\left|1-\frac{|k|}{2\pi}\right|\end{aligned} \tag{B.12}$$

To evaluate the final integral in (B.7), a similar approach to that leading to the evaluation (B.12) is adopted. Minor complications arise because of the need to modify the formula (B.11) for $l>k$. We find

$$\int_{-4\pi}^{4\pi} \frac{|k|}{8\pi} \log \left| \frac{2\pi - |l|}{2\pi - |k|} \right| \frac{\pi}{|k-l|} \frac{dl}{2\pi}$$
$$= \frac{|k|}{16\pi} \left\{ \text{dilog} \left(\frac{4\pi + |k|}{2\pi + |k|} \right) - \text{dilog} \left(\frac{|k|}{2\pi + |k|} \right) \right.$$
$$\left. - \log \left| 1 - \frac{|k|}{2\pi} \right| \log \left(\frac{4\pi + |k|}{|k|} \right) + g_1(k) \right\} \tag{B.13}$$

where g_1 is defined in (7.21). Substituting (B.8)–(B.13) in (B.7) gives the result (7.14).

ACKNOWLEDGMENTS

The work of PJF and DSM was supported by the Australian Research Council.

REFERENCES

1. K. Aomoto, Jacobi polynomials associated with Selberg's integral, *SIAM J. Math. Analysis* **18**:545–549 (1987).
2. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).
3. P. J. Forrester, Exact integral formulas and asymptotics for the correlations in the $1/r^2$ quantum many body system, *Phys. Lett. A* **179**:127–130 (1993).
4. P. J. Forrester, Recurrence equations for the computation of correlations in the $1/r^2$ quantum many body system, *J. Stat. Phys.* **72**:39–50 (1993).
5. P. J. Forrester, Addendum to Selberg correlation integrals and the $1/r^2$ quantum many body system, *Nucl. Phys. B* **416**:377–385 (1994).
6. P. J. Forrester and B. Jancovici, Exact and asymptotic formulas for overdamped Brownian dynamics, *Physica A* **238**:405–424 (1997).
7. P. J. Forrester, B. Jancovici, and E. R. Smith, The two-dimensional one-component plasma at $\Gamma = 2$: The semi-periodic strip, *J. Stat. Phys.* **31**:129–140 (1983).
8. P. J. Forrester and J. A. Zuk, Applications of the Dotsenko–Fateev integral in random-matrix models, *Nucl. Phys. B* **473**:616–630 (1996).
9. I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, 4th ed. (Academic Press, New York, 1980).
10. A. J. Guttmann, Indicators of solvability for lattice models, *Discr. Math.* **217**:167–189 (2000).
11. Z. N. C. Ha, Fractional statistics in one dimension: View from an exactly solvable model, *Nucl. Phys. B* **435**:604–636 (1995).
12. J. P. Hansen and I. R. McDonald, *Theory of Simple Liquids*, 2nd ed. (Academic Press, London, 1990).
13. B. Jancovici, Exact results for the two-dimensional one-component plasma, *Phys. Rev. Lett.* **46**:386–388 (1981).

14. B. Jancovici, Classical Coulomb systems near a plane wall. II, *J. Stat. Phys.* **29**:263–280 (1982).
15. K. W. J. Kadell, The Selberg–Jack symmetric functions, *Adv. Math.* **130**:33–102 (1997).
16. P. Kalinay, P. Markos, L. Samaj, and I. Travenec, The sixth-moment sum rule for the pair correlations of the two-dimensional one-component plasma: Exact results, *J. Stat. Phys.* **98**:639–666 (2000).
17. J. Kaneko, Selberg integrals and hypergeometric functions associated with Jack polynomials, *SIAM J. Math Anal.* **24**:1086–1110 (1993).
18. J. Lighthill, *Introduction to Fourier Analysis and Generalized Functions* (CUP, Cambridge, 1958).
19. I. G. Macdonald, *Hall Polynomials and Symmetric Functions*, 2nd ed. (Oxford University Press, Oxford, 1995).
20. M. L. Mehta, *Random Matrices*, 2nd ed. (Academic Press, New York, 1991).